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# The Peierls-Nabarro model as a limit of a Frenkel-Kontorova model

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## Abstract

We study a generalization of the fully overdamped Frenkel-Kontorova model in dimension  $n \geq 1$ . This model describes the evolution of the position of each atom in a crystal, and is mathematically given by an infinite system of coupled first order ODEs. We prove that for a suitable rescaling of this model, the solution converges to the solution of a Peierls-Nabarro model, which is a coupled system of two PDEs (typically an elliptic PDE in a domain with an evolution PDE on the boundary of the domain). This passage from the discrete model to a continuous model is done in the framework of viscosity solutions.

**MSC:** 49L25; 30E25

**Keywords:** Frenkel-Kontorova, Peierls-Nabarro, viscosity solutions, dislocations, discrete-continuous, boundary conditions.

## 1 Introduction

In this paper we are interested in two models describing the evolution of defects in crystals, called dislocations. These two models are the Frenkel-Kontorova model and the Peierls-Nabarro model. The Frenkel-Kontorova model is a discrete model which describes the evolution of the position of atoms in a crystal. On the contrary, the Peierls-Nabarro model is a continuous model where the dislocation is seen as a phase transition. The main goal of the paper is to show rigorously how the Peierls-Nabarro model can be obtained as a limit of the Frenkel-Kontorova model after a suitable rescaling.

### 1.1 Peierls-Nabarro model

Let us start to present the Peierls-Nabarro model. We set

$$\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\},$$

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and for a time  $0 < T \leq +\infty$ , we look for solutions  $u^0$  of the following system with  $\beta \geq 0$ :

$$\begin{cases} \beta u_t^0(x, t) = \Delta u^0(x, t) & (x, t) \in \Omega \times (0, T) \\ u_t^0(x, t) = F(u^0(x, t)) + \frac{\partial u^0}{\partial x_n}(x, t) & (x, t) \in \partial\Omega \times (0, T), \end{cases} \quad (1.1)$$

where the boundary  $\partial\Omega$  is defined by:

$$\partial\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n = 0\},$$

and the unknown  $u^0(x, t) \in \mathbb{R}$  is a scalar-valued function with the initial data

$$u^0(x, 0) = u_0(x) \quad \text{for} \quad \begin{cases} x \in \overline{\Omega} := \partial\Omega \cup \Omega & \text{if } \beta > 0 \\ x \in \partial\Omega & \text{if } \beta = 0. \end{cases} \quad (1.2)$$

We assume the following conditions on the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  and on the initial data  $u_0$  :

$$F \in W^{2,\infty}(\mathbb{R}) \quad \text{and} \quad \begin{cases} u_0 \in W^{3,\infty}(\overline{\Omega}) & \text{if } \beta > 0 \\ u_0 \in W^{3,\infty}(\partial\Omega) & \text{if } \beta = 0. \end{cases} \quad (1.3)$$

For simplicity we have taken a high regularity on the initial data, but this condition can be weakened. The classical Peierls-Nabarro model corresponds to the case  $\beta = 0$  with a 1-periodic function  $F$  (see Section 2). For a derivation and study of the Peierls-Nabarro model, we refer the reader to [22] (and [30, 28] for the original papers and [29] for a recent review by Nabarro on the Peierls-Nabarro model). Let us mention that a physical and numerical study of the evolution problem (1.1) has been treated in [27].

In this paper we consider the general case  $\beta \geq 0$ , since the case  $\beta = 0$  is natural in the Frenkel-Kontorova model (see Section 2), and mathematically the case  $\beta > 0$  is easier to study. In the special case  $\beta = 0$ , this problem can be reformulated (at least at a formal level) as a nonlocal evolution equation written on  $\partial\Omega$  in any dimension. See in particular [18] for such a reformulation in dimension  $n = 2$  and a limit of the Peierls-Nabarro model to the discrete dislocation dynamics (see also [26] for a homogenization result of the Peierls-Nabarro model). We refer the reader to [10] for the study of stationary solutions of our model when  $F = -W'$ .

## 1.2 Frenkel-Kontorova model

We now present the Frenkel-Kontorova model. This is a discrete model which contains a small scale  $\varepsilon > 0$  that can be seen as the order of the distance between atoms. We set the discrete analogue of  $\Omega$ :

$$\Omega^\varepsilon = (\varepsilon\mathbb{Z})^{n-1} \times \varepsilon(\mathbb{N} \setminus \{0\})$$

with discrete boundary

$$\partial\Omega^\varepsilon = (\varepsilon\mathbb{Z})^{n-1} \times \{0\},$$

and we set

$$\overline{\Omega}^\varepsilon = \Omega^\varepsilon \cup \partial\Omega^\varepsilon.$$

We look for solutions  $u^\varepsilon(x, t)$  with  $x \in \overline{\Omega}^\varepsilon$ ,  $t \geq 0$ , of the following system for  $\beta \geq 0$ :

$$\begin{cases} \beta u_t^\varepsilon(x, t) = \Delta^\varepsilon[u^\varepsilon](x, t), & (x, t) \in \Omega^\varepsilon \times (0, T) \\ u_t^\varepsilon(x, t) = F(u^\varepsilon(x, t)) + D^\varepsilon[u^\varepsilon](x, t), & (x, t) \in \partial\Omega^\varepsilon \times (0, T), \end{cases} \quad (1.4)$$

with initial data

$$u^\varepsilon(x, 0) = u_0(x) \quad \text{for} \quad \begin{cases} x \in \bar{\Omega}^\varepsilon & \text{if } \beta > 0 \\ x \in \partial\Omega^\varepsilon & \text{if } \beta = 0, \end{cases} \quad (1.5)$$

where we have used the following notation for the discrete analogue of  $\Delta$  and  $\frac{\partial}{\partial x_n}$  respectively:

$$\begin{cases} \Delta^\varepsilon[u^\varepsilon](x, t) = \frac{1}{\varepsilon^2} \sum_{y \in \mathbb{Z}^n, |y|=1} (u^\varepsilon(x + \varepsilon y, t) - u^\varepsilon(x, t)) \\ D^\varepsilon[u^\varepsilon](x, t) = \frac{1}{\varepsilon} \sum_{y \in \mathbb{Z}^n, |y|=1, y_n \geq 0} (u^\varepsilon(x + \varepsilon y, t) - u^\varepsilon(x, t)). \end{cases} \quad (1.6)$$

In this model, the function  $u^\varepsilon(x, t)$  describes the scalar analogue of the position of the atom of index  $x$  in the crystal. The classical fully overdamped Frenkel-Kontorova model is a one dimensional model. It corresponds to the particular case  $n = 2$  where  $u^\varepsilon(x, t) = 0$  if  $x \in \Omega^\varepsilon$ , and therefore only the  $u^\varepsilon(\varepsilon x, t)$  for  $x \in \mathbb{Z} \times \{0\}$  can be non trivial (we refer the reader to [9]). Remark that in the case  $\beta > 0$ , we can consider classical solutions using the Cauchy-Lipschitz theory, while at least in the case  $\beta = 0$ , it is convenient to deal with viscosity solutions (see Section 3).

More generally, Frenkel-Kontorova are also used for the description of vacancy defects at equilibrium, see [20]. See also [19], where the authors study the problem involving a dislocation inside the interphase between two identical lattices. Their model corresponds to our model (1.4) at the equilibrium with  $F = -W'$  where the potential  $W$  is a cosine function. For other 2D FK models, see [12, 13]. For homogenization results of some FK models, we refer the reader to [17].

### 1.3 Main result

Our main result is the following theorem which establishes rigorously (for the first time up to our knowledge) the link between the two famous physical models: the Frenkel-Kontorova model and the Peierls-Nabarro model.

**Theorem 1.1 (*Existence, uniqueness and convergence*)**

Let  $\varepsilon > 0$ ,  $\beta \geq 0$  and  $0 < T \leq +\infty$ . Under the condition (1.3) there exists a unique discrete viscosity solution  $u^\varepsilon$  of (1.4)-(1.5). Moreover, as  $\varepsilon \rightarrow 0$ , the sequence  $u^\varepsilon$  converges to the unique bounded viscosity solution  $u^0$  of (1.1)-(1.2). The convergence  $u^\varepsilon \rightarrow u^0$  has to be understood in the following sense: for any compact set  $K \subset \bar{\Omega} \times [0, T)$ , we have:

$$\|u^\varepsilon - u^0\|_{L^\infty(K \cap (\bar{\Omega}^\varepsilon \times [0, T)))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

A formal version of this result has been announced in [16]. We refer the reader to Section 3 for the definition of viscosity solutions. The proof of convergence is done in the framework of viscosity solutions, using the half-relaxed limits. The uniqueness of the limit  $u^0$  follows from a comparison principle that we prove for (1.1), using in particular a special test function introduced by Barles in [5]. Most of the difficulties arise here from the unboundedness of the domain, and in the non standard case  $\beta = 0$ , where the evolution equation is on the boundary of the domain instead of being in the interior of the domain. Let us notice that because system (1.4) can be seen as a discretization scheme of system (1.1), then Theorem 1.1 can be interpreted as a convergence result for such a scheme. In the literature on numerical analysis of finite difference schemes, we find other results of convergence for different equations (see for instance [8] and [7]). Let us mention that an estimate on the rate of convergence of  $u^\varepsilon$  to  $u^0$  is still an open question for our system.

**Remark 1.2** *From the proof of Theorem 1.1, it is easy to see that the result still holds true if  $F$  is replaced by a sequence  $(F^\varepsilon)_\varepsilon$  of functions which converges in  $W^{2,\infty}(\mathbb{R})$ . For such an example of application, see (2.3) at the end of Section 2; see also [16].*

## 1.4 Organization of the paper

This paper is organized as follows. In Section 2, we present the physical motivation to our problem. In Section 3, we present the definitions of viscosity solutions for the discrete and continuous problem (1.4)-(1.5) and (1.1)-(1.2) respectively. Section 4 is dedicated to construct uniform barriers of the solution  $u^\varepsilon$  of (1.4)-(1.5). Using those barriers, we prove in Section 5 the existence of a solution for the discrete problem (1.4)-(1.5). Section 6 is devoted to the proof of Theorem 1.1. Sections 7 and 8 are respectively devoted to the proofs of the comparison principle for the discrete problem (1.4)-(1.5) and the continuous problem (1.1)-(1.2), that were presented in Section 3 and used in the whole paper. Finally, in the Appendix, we give a convenient corollary of Ishii's lemma, that we use in Section 8.

## 2 Physical motivation

### 2.1 Geometrical description

We start to call  $(e_1, e_2, e_3)$  an orthonormal basis of the three dimensional space. In the corresponding coordinates, we consider a three-dimensional crystal where each atom is initially at the position of a node  $I$  of the lattice

$$\Lambda = \mathbb{Z}^2 \times \left( \frac{1}{2} + \mathbb{Z} \right).$$

This lattice is simply obtained by a translation along the vector  $\frac{1}{2}e_3$  of the lattice  $\mathbb{Z}^3$ , and will be more convenient for the derivation of our model. We assume that each atom  $I \in \Lambda$  of the crystal has the freedom to move to another position  $I + u_I e_2$  where  $u_I \in \mathbb{R}$  is a unidimensional displacement. In particular, we will be able to describe dislocations only with Burgers vectors that are multiples of the vector  $e_2$  (see [22] for an introduction to dislocations and a definition of the Burgers vector). Moreover we introduce the general notation

$$\bar{I} = (I', -I_3) \quad \text{for all } I = (I', I_3) \in \mathbb{Z}^2 \times \left( \frac{1}{2} + \mathbb{Z} \right) \quad \text{with } I' = (I_1, I_2) \in \mathbb{Z}^2$$

and will only consider antisymmetric displacements  $u_I$ , i.e. satisfying

$$u_{\bar{I}} = -u_I \quad \text{for all } I \in \Lambda. \tag{2.1}$$

We also introduce the discrete gradient

$$(\nabla^d u)_I = \begin{pmatrix} u_{I+e_1} - u_I \\ u_{I+e_2} - u_I \\ u_{I+e_3} - u_I \end{pmatrix}.$$

Generally, the core of a dislocation is localized where the discrete gradient is not small. In our model, screw and edge dislocations will be represented as follows.

### Screw dislocation

For  $I = (I_1, I_2, I_3)$ , we can consider a dislocation line parallel to the vector  $e_2$ , with a displacement  $u_I$  independent on  $I_2$ , i.e. such that  $u_I = s(I_1, I_3)$ . We will assume for instance that

$$\begin{cases} s(I_1, +\frac{1}{2}) = -s(I_1, -\frac{1}{2}) \rightarrow 0 & \text{as } I_1 \rightarrow -\infty \\ s(I_1, +\frac{1}{2}) = -s(I_1, -\frac{1}{2}) \rightarrow \frac{1}{2} & \text{as } I_1 \rightarrow +\infty. \end{cases}$$

Moreover, if there is no applied stress on the crystal, then it is reasonable to assume that

$$\text{dist} \left( (\nabla^d u)_I, \mathbb{Z}^3 \right) \rightarrow 0 \quad \text{as } |(I_1, I_3)| \rightarrow +\infty$$

which means that the crystal is perfect far from the core of the dislocation.

### Edge dislocation

For  $I = (I_1, I_2, I_3)$ , we can consider a dislocation line parallel to the vector  $e_1$ , with a displacement  $u_I$  independent on  $I_1$ , i.e.  $u_I = e(I_2, I_3)$ . We will assume for instance that

$$\begin{cases} e(I_2, +\frac{1}{2}) = -e(I_2, -\frac{1}{2}) \rightarrow 0 & \text{as } I_2 \rightarrow -\infty \\ e(I_2, +\frac{1}{2}) = -e(I_2, -\frac{1}{2}) \rightarrow \frac{1}{2} & \text{as } I_2 \rightarrow +\infty. \end{cases}$$

Moreover, if there is no applied stress on the crystal, then it is reasonable to assume that

$$\text{dist} \left( (\nabla^d u)_I, \mathbb{Z}^3 \right) \rightarrow 0 \quad \text{as } |(I_2, I_3)| \rightarrow +\infty.$$

Remark that this model can also describe more generally curved dislocations, which are neither screw, nor edge, but are mixed dislocations.

## 2.2 Energy of the crystal

We assume that each atom  $I$  is related to its nearest neighbors  $J$  by a nonlinear spring, whose force is derived from a smooth potential  $W_{IJ}$ . Then formally the full energy of the crystal after a displacement  $u = (u_I)_{I \in \Lambda}$  is

$$E(u) = \frac{1}{2} \sum_{I, J \in \Lambda, |I-J|=1} W_{IJ}(u_I - u_J).$$

We assume that the dislocation cores are only included in the double plane  $I_3 = \pm \frac{1}{2}$ . We also assume that

$$W_{IJ}(a) = \begin{cases} \varepsilon W(a) & \text{if } |I_3| = |J_3| = \frac{1}{2} \\ W(a) & \text{if } |I_3| \neq \frac{1}{2} \text{ or } |J_3| \neq \frac{1}{2}. \end{cases}$$

Here  $\varepsilon > 0$  is a small parameter. This means that the springs lying between the two planes  $I_3 = \pm \frac{1}{2}$  are very weak in comparison to the other springs. This will allow the core of the dislocation to spread out on the lattice as  $\varepsilon \rightarrow 0$  and will allow us to recover the Peierls-Nabarro model in this limit, after a suitable rescaling.

Notice that, to be compatible with the antisymmetry (2.1) of the crystal, it is reasonable to assume that

$$W(-a) = W(a) \quad \text{for all } a \in \mathbb{R}.$$

Recall that each line of atoms  $I^0 + \mathbb{Z}e_2$  only contains atoms of identical nature (i.e. surrounded by a similar configuration of springs). Therefore for  $u_I = s(I_1, I_3)$ , a lattice of atoms at the position

$I + u_I e_2$  and another lattice of atoms at the position  $I + u_I e_2 + k_{(I_1, I_3)} e_2$  with arbitrary  $k_{(I_1, I_3)} \in \mathbb{Z}$  are completely equivalent. Therefore their energy should be the same, and it is then natural to assume that the potential  $W$  is 1-periodic, i.e. satisfies

$$W(a+1) = W(a) \quad \text{for all } a \in \mathbb{R}.$$

We will also assume that non-deformed lattices minimize globally their energy, i.e. satisfy

$$W(\mathbb{Z}) = 0 \quad \text{and} \quad W > 0 \quad \text{on } \mathbb{R} \setminus \mathbb{Z}.$$

Let us now assume that a constant global shear stress is applied on the crystal such that it creates a global shear of the crystal in the direction  $e_2$  with respect to the coordinate  $I_3$ . This means that there exists a constant  $\tau \in \mathbb{R}$  such that

$$\text{dist} \left( (\nabla^d u)_I - \tau e_3, \mathbb{Z}^3 \right) \rightarrow 0$$

as  $I$  goes far away from the core of the dislocation. Remark that this assumption is compatible with the antisymmetry (2.1) of  $u$ . If there is no dislocations, we can simply take for instance  $u_I = \tau I_3$ .

### 2.3 Fully overdamped dynamics of the crystal

The natural dynamics should be given by Newton's law satisfied by each atom. This dynamics is very rich and for certain shear stress  $\tau$ , it is known (see [14, 21]) in 2D lattices that certain edge dislocations can propagate with constant mean velocity. This is due to the fact that part of the energy is lost by radiation of sound waves in the crystal. This phenomenon is similar to the effective drag force created by the surrounding fluid on a boat or an air plane. The resulting behaviour is a kind of dissipative dynamics that we modelise here by a fully overdamped dynamics of the crystal. See [23, 25, 24]) for a fundamental justification of overdamped type dynamics based on explicit computations in a 1D Hamiltonian model. For general physical justifications of the dissipative effects in the motion of dislocations, see also [22, 1].

We recall that we assume that the dislocation cores are only contained in the double plane  $I_3 = \pm \frac{1}{2}$ . For this reason, we will artificially distinguish the dynamics inside this double plane and outside this double plane. We consider the following fully overdamped dynamics (where the velocity of each atom is proportional to the force deriving from the energy) which is written formally as:

$$\alpha_I \dot{u}_I = -\nabla_{u_I} E(u) \quad \text{with} \quad \alpha_I = \begin{cases} 1 & \text{if } |I_3| = \frac{1}{2} \\ \alpha & \text{if } |I_3| \neq \frac{1}{2}, \end{cases} \quad (2.2)$$

for some constant  $\alpha \geq 0$ . Here  $\dot{u}_I$  denotes the time derivative of the displacement  $u_I$  and

$$\nabla_{u_I} E(u) := \sum_{J \in \Lambda, |J-I|=1} W'_{IJ}(u_I - u_J).$$

Remark in particular that this dynamics preserves the antisymmetry (2.1) of  $u$ . For slow motion of dislocations (i.e. small velocity with respect to the velocity of sound in the crystal), it is reasonable to assume that  $\alpha = 0$ , i.e. the lattice is instantaneously at the equilibrium outside the double plane  $I_3 = \pm \frac{1}{2}$ .

We also assume that the potential is harmonic close to its minima, i.e. satisfies

$$W(a) = \frac{a^2}{2} \quad \text{for } |a| \leq \delta < \frac{1}{2}$$

and assume that the strain is small enough in the crystal outside the double plane  $I_3 = \pm \frac{1}{2}$ , i.e. we assume that

$$|u_J - u_I| \leq \delta \quad \text{for any } J, I \in \Lambda \quad \text{such that } |J - I| = 1 \quad \text{and} \quad |I_3| \neq \frac{1}{2} \quad \text{or} \quad |J_3| \neq \frac{1}{2}.$$

This assumption allows us, in the region  $I_3 \neq \pm \frac{1}{2}$ , to consider forces that can be expressed linearly in terms of the displacement. We also set

$$\tau = \varepsilon \sigma \quad \text{for some } \sigma \in \mathbb{R} \quad \text{and} \quad v_I = u_I - \varepsilon \sigma I_3.$$

Using the antisymmetry of the solution (2.1), we deduce that we can rewrite the dynamics (2.2) with  $\alpha = 0$  as

$$\begin{cases} 0 = \sum_{J \in \Lambda, |J-I|=1} (v_J - v_I) & \text{for } I_3 > \frac{1}{2} \\ \dot{v}_I = \varepsilon \sigma - \varepsilon W'(2v_I + \varepsilon \sigma) + \sum_{J \in \Lambda, |J-I|=1, J_3 \geq \frac{1}{2}} (v_J - v_I) & \text{for } I_3 = \frac{1}{2}. \end{cases}$$

Then we see that

$$u^\varepsilon(x, t) = v_{(\frac{x}{\varepsilon} + \frac{1}{2})} \left( \frac{t}{\varepsilon} \right)$$

solves (1.4) with  $\beta = 0$  and  $F$  replaced by

$$F^\varepsilon(a) = \sigma - W'(2a + \varepsilon \sigma). \quad (2.3)$$

**Remark 2.1** *At the level of modeling, we could also consider more general lattices than  $\mathbb{Z}^n$  with more general nearest neighbors interactions, but this case is not covered by the result of Theorem 1.1 and would require a specific work.*

### 3 Viscosity solutions

In this section we present the notion of viscosity solutions and some of their properties for the discrete problem (1.4)-(1.5) and then for the continuous problem (1.1)-(1.2). For the classical notion of viscosity solutions, we refer the reader to [3], [4], [15].

#### 3.1 Viscosity solutions for the discrete problem

Before stating the definition of viscosity solutions for the discrete problem (1.4)-(1.5), we start by defining some terminology. Let  $\varepsilon, \delta > 0$  and  $0 < T \leq +\infty$ . Given a point  $P_0 = (x_0, t_0) \in \overline{\Omega}^\varepsilon \times [0, T)$ , the set  $\mathcal{N}_\delta^\varepsilon(t_0) \subset \overline{\Omega}^\varepsilon \times [0, T)$  is defined by:

$$\mathcal{N}_\delta^\varepsilon(P_0) = \{P = (y, t) \in \overline{\Omega}^\varepsilon \times [0, T); |y - x_0| \leq \varepsilon \text{ and } |t - t_0| < \delta\}. \quad (3.1)$$

The spaces  $USC(\overline{\Omega}^\varepsilon \times [0, T))$  and  $LSC(\overline{\Omega}^\varepsilon \times [0, T))$  are defined respectively by:

$$USC(\overline{\Omega}^\varepsilon \times [0, T)) = \{u \text{ defined on } \overline{\Omega}^\varepsilon \times [0, T); \forall x \in \overline{\Omega}^\varepsilon, u(x, \cdot) \in USC([0, T))\} \quad (3.2)$$



and

$$LSC(\overline{\Omega}^\varepsilon \times [0, T]) = \{u \text{ defined on } \overline{\Omega}^\varepsilon \times [0, T]; \forall x \in \overline{\Omega}^\varepsilon, u(x, \cdot) \in LSC([0, T])\}, \quad (3.3)$$

where  $USC([0, T])$  (resp.  $LSC([0, T])$ ) is the set of locally bounded upper (resp. lower) semi-continuous functions on  $[0, T]$ . In a similar manner, we define the space  $C^k(\overline{\Omega}^\varepsilon \times [0, T])$ ,  $k \in \mathbb{N}$ , by:

$$C^k(\overline{\Omega}^\varepsilon \times [0, T]) = \{u \text{ defined on } \overline{\Omega}^\varepsilon \times [0, T]; \forall x \in \overline{\Omega}^\varepsilon, u(x, \cdot) \in C^k([0, T])\}. \quad (3.4)$$

Next, given  $\beta \geq 0$ , we present the following definition:

**Definition 3.1 (Viscosity sub/super solutions)**

**1. Viscosity sub-solutions.** A function  $u \in USC(\overline{\Omega}^\varepsilon \times [0, T])$  is a viscosity sub-solution of problem (1.4)-(1.5) provided that:

$$(i) \ u(x, 0) \leq u_0(x) \text{ for } \begin{cases} x \in \overline{\Omega}^\varepsilon & \text{if } \beta > 0 \\ x \in \partial\Omega^\varepsilon & \text{if } \beta = 0, \end{cases}$$

(ii) for any  $\varphi \in C^1(\overline{\Omega}^\varepsilon \times [0, T])$ , if  $u - \varphi$  has a zero local maximum at a point  $P_0 = (x_0, t_0) \in \overline{\Omega}^\varepsilon \times [0, T]$  (i.e.  $\exists \delta > 0$  such that  $\forall P \in \mathcal{N}_\delta^\varepsilon(P_0)$ , we have  $(u - \varphi)(P) \leq (u - \varphi)(P_0) = 0$ ) then

$$\begin{cases} \beta \varphi_t(P_0) \leq \Delta^\varepsilon[\varphi](P_0) \\ \varphi_t(P_0) \leq F(\varphi(P_0)) + D^\varepsilon[\varphi](P_0) \end{cases} \quad \left| \begin{array}{ll} \text{for } P_0 \in \Omega^\varepsilon \times (0, T) & \text{if } \beta > 0 \\ \text{for } P_0 \in \Omega^\varepsilon \times [0, T] & \text{if } \beta = 0 \end{array} \right. \quad \text{for } P_0 \in \partial\Omega^\varepsilon \times (0, T). \quad (3.5)$$

**2. Viscosity super-solutions.** A function  $u \in LSC(\overline{\Omega}^\varepsilon \times [0, T])$  is a viscosity super-solution of problem (1.4)-(1.5) provided that:

$$(i) \ u(x, 0) \geq u_0(x) \text{ for } \begin{cases} x \in \overline{\Omega}^\varepsilon & \text{if } \beta > 0 \\ x \in \partial\Omega^\varepsilon & \text{if } \beta = 0, \end{cases}$$

(ii) for any  $\varphi \in C^1(\overline{\Omega}^\varepsilon \times [0, T])$ , if  $u - \varphi$  has a zero local minimum at a point  $P_0 = (x_0, t_0) \in \overline{\Omega}^\varepsilon \times [0, T]$  (i.e.  $\exists \delta > 0$  such that  $\forall P \in \mathcal{N}_\delta^\varepsilon(P_0)$ , we have  $(u - \varphi)(P) \geq (u - \varphi)(P_0) = 0$ ) then

$$\begin{cases} \beta \varphi_t(P_0) \geq \Delta^\varepsilon[\varphi](P_0) \\ \varphi_t(P_0) \geq F(\varphi(P_0)) + D^\varepsilon[\varphi](P_0) \end{cases} \quad \left| \begin{array}{ll} \text{for } P_0 \in \Omega^\varepsilon \times (0, T) & \text{if } \beta > 0 \\ \text{for } P_0 \in \Omega^\varepsilon \times [0, T] & \text{if } \beta = 0 \end{array} \right. \quad \text{for } P_0 \in \partial\Omega^\varepsilon \times (0, T). \quad (3.6)$$

**3. Viscosity solutions.** A function  $u \in C^0(\overline{\Omega}^\varepsilon \times [0, T])$  is a viscosity solution of problem (1.4)-(1.5) if it is a viscosity sub- and super-solution of (1.4)-(1.5).

**Definition 3.2** We say that  $u$  is a viscosity sub-solution (resp. super-solution) of (1.4) if  $u$  only satisfies (ii) in the Definition 3.1.

**Remark 3.3** Note that a function  $u$  is a viscosity subsolution of (1.4) if it satisfies the viscosity inequalities on  $\overline{\Omega}^\varepsilon \times (0, T)$  if  $\beta > 0$  and on  $(\overline{\Omega}^\varepsilon \times (0, T)) \cup (\Omega^\varepsilon \times \{0\})$  if  $\beta = 0$ .

**Theorem 3.4 (Comparison principle in the discrete case)**

Let  $u \in USC(\overline{\Omega}^\varepsilon \times [0, T])$  (resp.  $v \in LSC(\overline{\Omega}^\varepsilon \times [0, T])$ ) be a viscosity sub-solution (resp. super-solution) for the problem (1.4)-(1.5) where  $0 < T < \infty$ , such that  $\|u\|_{L^\infty(\overline{\Omega}^\varepsilon \times [0, T])}, \|v\|_{L^\infty(\overline{\Omega}^\varepsilon \times [0, T])} < +\infty$ . Then

$$u(x, t) \leq v(x, t) \quad \text{for all } (x, t) \in \overline{\Omega}^\varepsilon \times [0, T].$$

Theorem 3.4 will be proved in Section 7.

### 3.2 Viscosity solutions for the continuous problem

Similarly as in subsection 3.1 we denote by  $USC(\overline{\Omega} \times [0, T])$  (resp.  $LSC(\overline{\Omega} \times [0, T])$ ) the set of locally bounded upper (resp. lower) semi-continuous functions on  $\overline{\Omega} \times [0, T]$ . We also denote by  $C^k(\overline{\Omega} \times [0, T])$  the classical set of continuously  $k$ -differentiable functions on  $\overline{\Omega} \times [0, T]$ .

#### Definition 3.5 (*Viscosity sub/super solutions*)

**1. Viscosity sub-solutions.** A function  $u \in USC(\overline{\Omega} \times [0, T])$  is a viscosity sub-solution of problem (1.1)-(1.2) provided that:

$$(i) \ u \leq u_0 \text{ on } \begin{cases} \overline{\Omega} \times \{0\} & \text{if } \beta > 0 \\ \partial\Omega \times \{0\} & \text{if } \beta = 0. \end{cases}$$

(ii) for any  $\varphi \in C^2(\overline{\Omega} \times [0, T])$ , if  $u - \varphi$  has a zero local maximum at a point  $P_0 = (x_0, t_0) \in \overline{\Omega} \times [0, T]$  then

$$\begin{cases} \beta\varphi_t(P_0) \leq \Delta\varphi(P_0) & \left| \begin{array}{ll} \text{for } P_0 \in \Omega \times (0, T) & \text{if } \beta > 0 \\ \text{for } P_0 \in \Omega \times [0, T] & \text{if } \beta = 0 \end{array} \right. \\ \min \left\{ \beta\varphi_t(P_0) - \Delta\varphi(P_0), \varphi_t(P_0) - F(\varphi(P_0)) - \frac{\partial\varphi}{\partial x_n}(P_0) \right\} \leq 0 & \text{for } P_0 \in \partial\Omega \times (0, T). \end{cases} \quad (3.7)$$

**2. Viscosity super-solutions.** A function  $u \in LSC(\overline{\Omega} \times [0, T])$  is a viscosity super-solution of problem (1.1)-(1.2) provided that:

$$(i) \ u \geq u_0 \text{ on } \begin{cases} \overline{\Omega} \times \{0\} & \text{if } \beta > 0 \\ \partial\Omega \times \{0\} & \text{if } \beta = 0. \end{cases}$$

(ii) for any  $\varphi \in C^2(\overline{\Omega} \times [0, T])$ , if  $u - \varphi$  has a zero local minimum at a point  $P_0 = (x_0, t_0) \in \overline{\Omega} \times [0, T]$  then

$$\begin{cases} \beta\varphi_t(P_0) \geq \Delta\varphi(P_0) & \left| \begin{array}{ll} \text{for } P_0 \in \Omega \times (0, T) & \text{if } \beta > 0 \\ \text{for } P_0 \in \Omega \times [0, T] & \text{if } \beta = 0 \end{array} \right. \\ \max \left\{ \beta\varphi_t(P_0) - \Delta\varphi(P_0), \varphi_t(P_0) - F(\varphi(P_0)) - \frac{\partial\varphi}{\partial x_n}(P_0) \right\} \geq 0 & \text{for } P_0 \in \partial\Omega \times (0, T). \end{cases} \quad (3.8)$$

**3. Viscosity solutions.** A function  $u \in C^0(\overline{\Omega} \times [0, T])$  is a viscosity solution of problem (1.1)-(1.2) if it is a viscosity sub- and super-solution of (1.1)-(1.2).

#### Theorem 3.6 (*Comparison principle in the continuous case*)

Let  $u \in USC(\overline{\Omega} \times [0, T])$  (resp.  $v \in LSC(\overline{\Omega} \times [0, T])$ ) be a viscosity sub-solution (resp. super-solution) of problem (1.1)-(1.2) where  $0 < T < \infty$ , such that  $\|u\|_{L^\infty(\overline{\Omega} \times [0, T])}, \|v\|_{L^\infty(\overline{\Omega} \times [0, T])} < +\infty$ . Then

$$u \leq v \quad \text{on } \overline{\Omega} \times [0, T]$$

This theorem will be proved in section 8.

**Remark 3.7** Remark that in the case  $\beta = 0$ , the problem can be reformulated (at least for smooth solutions) as a nonlocal evolution equation written on the boundary  $\partial\Omega$ . See for instance the work [11] on the relation between fractional Laplacian and harmonic extensions. Note that there is also a viscosity theory for nonlocal operators (see [6]).

## 4 Construction of barriers

This section is devoted to the construction of barriers for the solution  $u^\varepsilon$  of (1.4)-(1.5) for all  $\beta \geq 0$ .

#### 4.1 Discrete harmonic extension

Here we construct the discrete harmonic extension of  $u_0$  on  $\overline{\Omega}^\varepsilon$ , i.e. we prove the existence of a solution of the following problem

$$\begin{cases} -\Delta^1[u_0^D] = 0, & \text{in } \Omega^1 \\ u_0^D = u_0, & \text{on } \partial\Omega^1, \end{cases} \quad (4.1)$$

where  $\Delta^1 = \Delta^\varepsilon$  for  $\varepsilon = 1$  is defined in (1.6). We note here that, for the sake of simplicity, we have taken  $\varepsilon = 1$ , while otherwise it can be treated in the same way (or simply deduced by rescaling). We will say that a function  $u$  is a sub-solution (resp. super-solution) of (4.1) if the symbol “=” in (4.1) is replaced by the symbol “ $\leq$ ” (resp. “ $\geq$ ”).

**Lemma 4.1** *If  $u_0 \in L^\infty(\partial\Omega^1)$ , then there exists a unique (viscosity) solution of (4.1) on  $\overline{\Omega}^1$ .*

**Proof.**

**Step 1: Existence.**

Define

$$\overline{u} = \sup_{\partial\Omega^1} u_0 \quad \text{and} \quad \underline{u} = \inf_{\partial\Omega^1} u_0,$$

then  $\underline{u}$  (resp.  $\overline{u}$ ) is a sub- (resp. super-) solution of (4.1). Consider now the set

$$\mathcal{S} = \left\{ u : \overline{\Omega}^1 \rightarrow \mathbb{R}, \text{ subsolution of (4.1) s.t. } \underline{u} \leq u \leq \overline{u} \right\},$$

and in order to use Perron's method, we define

$$u_0^D := \sup_{u \in \mathcal{S}} u.$$

Using the fact that the maximum of two sub-solutions is a sub-solution, it is possible to show that  $u_0^D$  is a sub-solution of (4.1). Hence, it remains to prove that  $u_0^D$  is a super-solution for (4.1). Suppose that there exists  $x_0 \in \overline{\Omega}^1$  such that

$$\begin{cases} -\Delta^1[u_0^D](x_0) < 0, & \text{if } x_0 \in \Omega^1 \\ \text{or} \\ u_0^D(x_0) < u_0(x_0), & \text{if } x_0 \in \partial\Omega^1. \end{cases}$$

Then we construct a sub-solution  $w \in \mathcal{S}$  such that  $w > u_0^D$  at  $x_0$ . Two cases are considered.

**Case  $-\Delta^1[u_0^D](x_0) < 0$  for  $x_0 \in \Omega^1$ .** Let  $w$  be defined by

$$w(x) = \begin{cases} u_0^D(x) & \text{if } x \neq x_0, \\ u_0^D(x_0) + \frac{1}{2n} \Delta^1[u_0^D](x_0) & \text{if } x = x_0. \end{cases}$$

We check that  $\underline{u} \leq w \leq \overline{u}$ , and we compute

$$\Delta^1[w](x) = \begin{cases} \geq \Delta^1[u_0^D](x) \geq 0 & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0, \end{cases}$$

where we have used the fact that  $w \geq u_0^D$  to deduce the inequality. Therefore, we have  $w \in \mathcal{S}$  and  $w(x_0) > u_0^D(x_0)$  which is a contradiction.

**Case  $u_0^D(\mathbf{x}_0) < u_0(\mathbf{x}_0)$  for  $\mathbf{x}_0 \in \partial\Omega^1$ .** Let  $w$  be defined by

$$w(x) = \begin{cases} u_0^D(x) & \text{if } x \neq x_0, \\ u_0(x_0) & \text{if } x = x_0. \end{cases}$$

By construction, we have  $\underline{u} \leq w \leq \bar{u}$ . Moreover, as  $w \geq u_0^D$  it follows that  $\Delta^1[w] \geq \Delta^1[u_0^D] \geq 0$  on  $\Omega^1$ . Therefore  $w \in \mathcal{S}$  and  $w(x_0) = u_0(x_0) > u_0^D(x_0)$ ; this implies a contradiction.

## Step 2: Uniqueness.

We simply adapt case 1 i) of the proof of the comparison principle (Theorem 3.4), given in Section 7.  $\square$

## 4.2 Continuous harmonic extension

For the case  $\beta = 0$ , we need to consider the harmonic "extension"  $u_0^c : \overline{\Omega} \rightarrow \mathbb{R}$  of the initial data  $u_0$ . This function is the solution of

$$\begin{cases} -\Delta u_0^c = 0 & \text{on } \Omega \\ u_0^c = u_0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

This section is devoted to recall the existence of the continuous harmonic extension and to show some of its properties. Let  $z = (z', z_n) \in \Omega$  where  $z'$  is identified to an element of  $\partial\Omega$  and  $z_n > 0$ , and let  $H$  be a new function defined by

$$H(z', z_n) := \frac{2z_n}{\omega_n(z_n^2 + z'^2)^{n/2}}, \quad (4.3)$$

where  $\omega_n$  is the measure of the  $(n-1)$ -dimensional sphere in  $\mathbb{R}^n$ . Now, we define  $u_0^c$  by

$$u_0^c(x) := \int_{\partial\Omega} H(x' - z', x_n) u_0(z') dz' \quad \text{for all } x = (x', x_n) \in \mathbb{R}^{n-1} \times (0, +\infty). \quad (4.4)$$

Note that  $H(x' - z', x_n)$  is the Poisson kernel. We have the following three lemmas:

### Lemma 4.2 (*Existence of a continuous harmonic extension*)

If  $u_0(z')$  is bounded and continuous for  $z' \in \partial\Omega$ , then the function  $u_0^c(x)$  defined by (4.4) belongs to  $C^\infty(\Omega)$  and is harmonic in  $\Omega$  and extends continuously to  $\overline{\Omega}$  such that  $u_0^c = u_0$  on  $\partial\Omega$ .

**Proof.** See [2, Section 7.3, p. 129].  $\square$

### Lemma 4.3 (*Estimate on the harmonic extension*)

If  $u_0 \in W^{3,\infty}(\partial\Omega)$ , then we have

$$|Du_0^c(x)| \leq \frac{C}{1+x_n}, \quad |D^2u_0^c(x)| \leq \frac{C}{1+x_n^2} \quad \text{for all } x = (x', x_n) \in \mathbb{R}^{n-1} \times (0, +\infty). \quad (4.5)$$

**Proof.** Since  $u_0$  is bounded, then by adding an appropriate constant to  $u_0$  we can always assume, without loss of generality, that  $u_0 \geq 0$ . From (4.4) and from the following property (see [2, p.126])

$$\int_{\partial\Omega} H(x' - z', x_n) dz' = 1 \quad \text{for all } x = (x', x_n) \in \overline{\Omega},$$

we have:

$$|u_0^c| \leq \int_{\partial\Omega} H(x' - z', x_n) |u_0(z')| dz' \leq \|u_0\|_{L^\infty} \int_{\partial\Omega} H(x' - z', x_n) dz' = \|u_0\|_{L^\infty},$$

and then

$$\|u_0^c\|_{L^\infty} \leq \|u_0\|_{L^\infty}. \quad (4.6)$$

Next, for  $x = (x', x_n) \in \mathbb{R}^{n-1} \times (0, +\infty)$ , using (4.3) and (4.4), we have, on the one hand with  $\mu = \frac{2}{\omega_n}$ :

$$\frac{1}{\mu} \frac{\partial u_0^c}{\partial x_n} = \int_{\partial\Omega} \frac{u_0(z')}{(x_n^2 + (x' - z')^2)^{n/2}} dz' - \int_{\partial\Omega} \frac{nx_n^2 u_0(z')}{(x_n^2 + (x' - z')^2)^{n/2+1}} dz', \quad (4.7)$$

then we obtain

$$\left| \frac{\partial u_0^c}{\partial x_n} \right| \leq \frac{\|u_0\|_{L^\infty}}{x_n} + n \frac{\|u_0\|_{L^\infty}}{x_n} \leq \frac{C}{x_n}$$

where we have used (4.6). On the other hand, we have:

$$\frac{1}{\mu} \frac{\partial u_0^c}{\partial x'} = -n \int_{\partial\Omega} \frac{x_n u_0(z') (x' - z')}{(x_n^2 + (x' - z')^2)^{n/2+1}} dz', \quad (4.8)$$

then, using Young's inequality:

$$x_n |x' - z'| \leq \frac{1}{2} (x_n^2 + (x' - z')^2), \quad (4.9)$$

we conclude that

$$\left| \frac{\partial u_0^c}{\partial x'} \right| \leq \frac{n\mu}{2} \int_{\partial\Omega} \frac{u_0(z')}{(x_n^2 + (x' - z')^2)^{n/2}} dz' \leq \frac{n}{2x_n} \|u_0\|_{L^\infty} \leq \frac{C}{x_n}.$$

Therefore

$$|Du_0^c(x)| \leq \frac{C}{x_n}.$$

In order to prove the second inequality in (4.5), we differentiate (4.7) and (4.8), we get

$$\frac{1}{\mu} \frac{\partial^2 u_0^c}{\partial x' \partial x'} = \int_{\partial\Omega} \frac{u_0(z') [-nx_n I']}{(x_n^2 + (x' - z')^2)^{n/2+1}} dz' + \int_{\partial\Omega} \frac{x_n u_0(z') (n)(n+2)(x' - z') \otimes (x' - z')}{(x_n^2 + (x' - z')^2)^{n/2+2}} dz',$$

where  $I'$  is the identity matrix of  $\mathbb{R}^{n-1}$ , then, as  $1/(x_n^2 + (x' - z')^2) \leq 1/x_n^2$ , we obtain

$$\frac{1}{\mu} \left| \frac{\partial^2 u_0^c}{\partial x' \partial x'} \right| \leq n \int_{\partial\Omega} \frac{u_0(z') x_n}{(x_n^2 + (x' - z')^2)^{n/2} x_n^2} dz' + n(n+2) \int_{\partial\Omega} \frac{u_0(z') x_n (x' - z')^2}{(x_n^2 + (x' - z')^2)^{n/2} (x_n^2 + (x' - z')^2)^2} dz'.$$

Thus, using the following inequality

$$\frac{(x' - z')^2}{(x_n^2 + (x' - z')^2)^2} \leq \frac{1}{x_n^2},$$

we deduce that

$$\left| \frac{\partial^2 u_0^c}{\partial x' \partial x'} \right| \leq \frac{C}{x_n^2}. \quad (4.10)$$

Moreover, we have

$$\begin{aligned} \frac{1}{\mu} \frac{\partial^2 u_0^c}{\partial x' \partial x_n} &= \int_{\partial\Omega} \frac{u_0(z')(-n)(x' - z')}{(x_n^2 + (x' - z')^2)^{n/2+1}} dz' + \int_{\partial\Omega} \frac{u_0(z')x_n^2(n)(n+2)(x' - z')}{(x_n^2 + (x' - z')^2)^{n/2+2}} dz', \\ &= \frac{1}{x_n^2} \int_{\partial\Omega} \frac{u_0(z')x_n(-n)x_n(x' - z')}{(x_n^2 + (x' - z')^2)^{n/2}(x_n^2 + (x' - z')^2)} dz' + \int_{\partial\Omega} \frac{u_0(z')x_n^2(n)(n+2)(x' - z')}{(x_n^2 + (x' - z')^2)^{n/2+2}} dz'. \end{aligned}$$

Then, using (4.6) and (4.9), we infer that

$$\left| \frac{\partial^2 u_0^c}{\partial x' \partial x_n} \right| \leq \frac{C}{x_n^2}.$$

Next since,

$$\frac{1}{\mu} \frac{\partial^2 u_0^c}{\partial x_n \partial x_n} = -3n \int_{\partial\Omega} \frac{u_0(z')x_n}{(x_n^2 + (x' - z')^2)^{n/2+1}} dz' + n(n+2) \int_{\partial\Omega} \frac{u_0(z')x_n^3}{(x_n^2 + (x' - z')^2)^{n/2+2}} dz',$$

we use similar arguments as above in order to obtain

$$\left| \frac{\partial^2 u_0^c}{\partial x_n \partial x_n} \right| \leq \frac{C}{x_n^2}.$$

Notice that this last inequality also follows from (4.10) joint to the harmonicity of  $u_0^c$ .

Finally, we use Schauder's estimate near the boundary  $\partial\Omega$  :

$$|u_0^c|_{C^{2,\alpha}(\overline{B_1(x',0) \cap \Omega})} \leq C (|\Delta u_0^c|_{C^\alpha(B_2(x',0) \cap \Omega)} + |u_0|_{C^{2,\alpha}(B_2(x',0) \cap \partial\Omega)}) = C |u_0|_{C^{2,\alpha}(B_2(x',0) \cap \partial\Omega)},$$

for all  $x' \in \mathbb{R}^{n-1}$ , where  $C^\alpha$  and  $C^{2,\alpha}$ ,  $\alpha \in (0,1)$ , are the Hölder spaces of order  $\alpha$  and  $2 + \alpha$ , respectively. The ball  $B_r(x',0)$ ,  $r = 1,2$ , stands for the ball of center  $(x',0)$  and radius  $r$ . We finally conclude that:

$$\sup_{x' \in \mathbb{R}^{n-1}} |u_0^c|_{C^{2,\alpha}(\overline{B_1(x',0) \cap \Omega})} \leq C \sup_{x' \in \mathbb{R}^{n-1}} |u_0|_{C^{2,\alpha}(B_2(x',0) \cap \partial\Omega)},$$

and the result follows.  $\square$

**Lemma 4.4 ( $\varepsilon$ -uniform bound)**

Under the assumption (1.3) we have

$$|D^\varepsilon[u_0^c](x)| \leq \frac{C_1}{1+x_n}, \quad |\Delta^\varepsilon[u_0^c](x)| \leq \frac{C_1}{1+x_n^2} \quad \text{for all } x = (x', x_n) \in \Omega^\varepsilon, \quad (4.11)$$

where  $C_1 > 0$  is a positive constant independent on  $\varepsilon$ .

**Proof.** Let us set  $\rho_n^- = 0$  and  $\rho_i^\alpha = 1$  otherwise. Then using Taylor's expansion, we have for  $x \in \Omega^\varepsilon$ :

$$\varepsilon D^\varepsilon[u_0^c](x) := \sum_{\pm} \sum_{i=1}^n \rho_i^\pm (u_0^c(x \pm \varepsilon e_i) - u_0^c(x)) = \sum_{\pm} \sum_{i=1}^n \rho_i^\pm \int_0^1 dt Du_0^c(x \pm \varepsilon t e_i)(\pm \varepsilon e_i),$$

where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$  is the unit vector in  $\mathbb{R}^n$  with respect to the  $i$  component. Then, using (4.5), we get

$$|D^\varepsilon[u_0^c](x)| \leq (2n-1) \sup_{1 \leq i \leq n} \int_0^1 dt \rho_i^\pm |Du_0^c(x \pm \varepsilon t e_i)| \leq \frac{C_1}{1+x_n}.$$

$\pm$

Then, in order to terminate the proof, it is sufficient to write  $\Delta^\varepsilon[u_0^D]$  using Taylor's expansion as

$$\begin{aligned}
\varepsilon^2 \Delta^\varepsilon[u_0^c](x) &:= \sum_{\pm} \sum_{i=1}^n (u_0^c(x \pm \varepsilon e_i) - u_0^c(x)) \\
&= \sum_{\pm} \sum_{i=1}^n \left( \pm \varepsilon e_i \nabla u_0^c(x) + \int_0^1 dt \int_0^t ds D^2 u_0^c(x \pm \varepsilon s e_i) (\pm \varepsilon e_i) (\pm \varepsilon e_i) \right) \\
&= \varepsilon^2 \sum_{\pm} \sum_{i=1}^n \int_0^1 dt \int_0^t ds D^2 u_0^c(x \pm \varepsilon s e_i) \cdot (e_i, e_i).
\end{aligned} \tag{4.12}$$

Thus, thanks to (4.5), we finally obtain:

$$|\Delta^\varepsilon[u_0^c](x)| \leq 2n \sup_{\substack{1 \leq i \leq n \\ \pm}} \int_0^1 dt \int_0^t ds |D^2 u_0^c(x \pm \varepsilon s e_i)| \leq \frac{C_1}{1+x_n^2},$$

and the proof is done.  $\square$

### 4.3 Uniform barriers for $\beta \geq 0$

In this subsection, we show uniform barriers for the solution  $u^\varepsilon$  of (1.4)-(1.5) for all  $\beta \geq 0$  in the special case where  $u_0 = u_0^c$ .

**Proposition 4.5** (*Uniform barriers in  $\varepsilon$  for all  $\beta \geq 0$  for  $u_0 = u_0^c$* )

Under assumption (1.3), there exists a constant  $C > 0$  independent on  $\varepsilon > 0$ ,  $\beta \geq 0$  and  $T$  such that if

$$\left\{ \begin{array}{l} \bar{u}^+(x, t) := u_0^c(x) + C(\sqrt{1+x_n} - 1 + t) \\ \bar{u}^-(x, t) := u_0^c(x) - C(\sqrt{1+x_n} - 1 - t) \end{array} \right\} \quad \text{for } (x, t) \in \bar{\Omega}^\varepsilon \times [0, T], \tag{4.13}$$

then  $\bar{u}^+$  (resp.  $\bar{u}^-$ ) is a supersolution (resp. subsolution) of (1.4)-(1.5). However, if  $u^\varepsilon$  is a bounded viscosity solution of (1.4)-(1.5), then we have

$$\bar{u}^- \leq u^\varepsilon \leq \bar{u}^+ \quad \text{on } \bar{\Omega}^\varepsilon \times [0, T], \tag{4.14}$$

and moreover:

$$|u^\varepsilon(x, t)| \leq \|u_0\|_{L^\infty} + t\|F\|_{L^\infty} \quad \text{for } (x, t) \in \bar{\Omega}^\varepsilon \times [0, T].$$

**Proof.**

**Step 1: sub/supersolution property**

We first check that  $\bar{u}^+$  is a super-solution of (1.4)-(1.5). Indeed,

- If  $(x, t) \in \bar{\Omega}^\varepsilon \times \{0\}$ , then  $\bar{u}^+(x, t) \geq u_0^c(x) = u_0(x)$ .
- If  $(x, t) \in \partial\Omega^\varepsilon \times (0, T)$ , then we take  $C > 0$  such that

$$C \geq F(\bar{u}^+(x, t)) + D^\varepsilon[u_0^c](x) + CD^\varepsilon[\sqrt{1+x_n}] \iff \bar{u}_t^+(x, t) \geq F(\bar{u}^+(x, t)) + D^\varepsilon[\bar{u}^+](x, t).$$

Using the fact that  $D^\varepsilon[\sqrt{1+x_n}](x_n = 0) \leq \frac{1}{2}$ , we see that such a constant  $C$  exists.

- If  $(x, t) \in \Omega^\varepsilon \times [0, T]$ , then we can use Lemma 4.4. We get

$$\Delta^\varepsilon[u_0^c](x) \leq \frac{C_1}{1+x_n^2} \leq \frac{C}{4(1+x_n)^{3/2}},$$

where the last inequality is true for  $C > 0$  large enough. Moreover, by repeating the same computations as in (4.12), we easily get with  $f(a) = \sqrt{1+a}$ :

$$\Delta^\varepsilon[\sqrt{1+x_n}] = \int_0^1 dt \int_0^t ds \sum_{\pm} f''(x_n \pm \varepsilon s) \leq \int_0^1 dt \int_0^t ds .2.f''(x_n) \leq f''(x_n) = -\frac{1}{4(1+x_n)^{3/2}}, \quad (4.15)$$

where we have used the concavity of the function  $f''$ . Then we have

$$\Delta^\varepsilon[u_0^c](x) \leq \frac{C}{4(1+x_n)^{3/2}} \leq -C\Delta^\varepsilon[\sqrt{1+x_n}],$$

hence

$$\beta \bar{u}_t^+(x, t) = \beta C \geq 0 \geq \Delta^\varepsilon[\bar{u}^+](x, t).$$

As a consequence, we finally deduce that  $\bar{u}^+$  is a super-solution. In a similar way, by taking in addition the following assumption on  $C$ :

$$C \geq -F(\bar{u}^-(x, t)) - D^\varepsilon[u_0^c](x) + CD^\varepsilon[\sqrt{1+x_n}]$$

we can prove that  $\bar{u}^-$  is a sub-solution.

#### Step 2: bounds on $u^\varepsilon$

Let us call  $\bar{v}(x, t) := \|u_0\|_{L^\infty} + t\|F\|_{L^\infty}$ . It is easy to check that  $\bar{v}$  is a super-solution of (1.4)-(1.5). Then

$$\min(\bar{u}^+, \bar{v})$$

is still a super-solution and is bounded. Therefore, if  $u^\varepsilon$  is a bounded viscosity solution of (1.4)-(1.5), we can then apply the comparison principle (Theorem 3.4) to conclude that

$$u^\varepsilon \leq \min(\bar{u}^+, \bar{v}).$$

This shows the upper bounds. For the lower bounds, we proceed similarly with  $\max(\bar{u}^-, -\bar{v})$ .  $\square$

#### 4.4 Barriers for $\beta > 0$

We have the following

##### Proposition 4.6 (*Barriers for $\beta > 0$* )

*Under the assumptions of Theorem 1.1, for every  $\beta > 0$ , there exists a constant  $C_\beta > 0$  such that for all  $\varepsilon$ , if  $u^\varepsilon$  is a bounded viscosity solution of (1.4)-(1.5), then we have*

$$|u^\varepsilon(x, t) - u_0(x)| \leq tC_\beta \quad \text{for } (x, t) \in \bar{\Omega}^\varepsilon \times [0, T].$$

##### Proof.

Using in particular the fact that  $F$  is bounded and that  $u_0 \in W^{2,\infty}(\bar{\Omega})$ , we simply check that  $u_0(x) + tC_\beta$  is a super-solution for a suitable constant  $C_\beta > 0$  and apply the comparison principle. We proceed similarly for sub-solutions  $u_0(x) - tC_\beta$ .  $\square$



## 5 Existence and uniqueness of a solution for the discrete problem

The aim of this section is to prove the existence of solutions of problem (1.4)-(1.5). Cauchy-Lipschitz method is the main tool used to prove the existence of solutions for  $\beta > 0$ , while in the case  $\beta = 0$  we need barriers to prove the existence.

**Theorem 5.1** (*Existence and uniqueness,  $\beta > 0$* )

If  $u_0$  and  $F$  satisfy (1.3) and  $\beta > 0$ , then there exists a unique bounded solution  $u^{\beta,\varepsilon} \in C^1(\overline{\Omega}^\varepsilon \times [0, T])$  of problem (1.4)-(1.5).

**Proof.** The proof is done using the classical Cauchy-Lipschitz theorem. Let  $B := L^\infty(\overline{\Omega}^\varepsilon)$  be the Banach space with the norm

$$\|u\|_B := \sup_{x \in \overline{\Omega}^\varepsilon} |u(x)|, \quad \text{for every } u \in B,$$

and  $\mathcal{F} : B \longrightarrow B$  be the map defined, for every  $u \in B$  and  $x \in \overline{\Omega}^\varepsilon$ , by

$$\mathcal{F}[u](x) := \begin{cases} \frac{1}{\beta} \Delta^\varepsilon[u](x), & x \in \Omega^\varepsilon, \\ F(u(x)) + D^\varepsilon[u](x), & x \in \partial\Omega^\varepsilon \end{cases}$$

where  $\Delta^\varepsilon[u](x)$  and  $D^\varepsilon[u](x)$  are defined as in (1.6), dropping the variable  $t$  on both sides of (1.6). Then, for every  $u, v \in B$  and  $x \in \overline{\Omega}^\varepsilon$ , we have two cases; either  $x \in \Omega^\varepsilon$ , and hence we obtain

$$\begin{aligned} \|\mathcal{F}[u](x) - \mathcal{F}[v](x)\|_B &\leq \left\| \frac{1}{\beta} \Delta^\varepsilon[u - v](x) \right\|_B \\ &\leq \frac{4n}{\beta \varepsilon^2} \|u - v\|_B, \end{aligned}$$

or  $x \in \partial\Omega^\varepsilon$ , then:

$$\begin{aligned} \|\mathcal{F}[u](x) - \mathcal{F}[v](x)\|_B &\leq \|D^\varepsilon[u - v](x) + |F(u(x)) - F(v(x))|\|_B \\ &\leq \left( \frac{2(2n-1)}{\varepsilon} + \|F'\|_{L^\infty(\mathbb{R})} \right) \|u - v\|_B. \end{aligned}$$

In all cases we conclude that  $\mathcal{F}$  is globally Lipschitz continuous. By the Cauchy-Lipschitz theorem, we get the existence and uniqueness of a solution  $u^{\beta,\varepsilon} \in C^1(\overline{\Omega}^\varepsilon \times [0, T])$  satisfying

$$u_t^{\beta,\varepsilon}(\cdot, t) = \mathcal{F}[u^{\beta,\varepsilon}](\cdot, t),$$

such that  $u^{\beta,\varepsilon}(\cdot, 0) = u_0(\cdot)$ . □

**Theorem 5.2** (*Existence and uniqueness,  $\beta = 0$* )

If  $u_0$  and  $F$  satisfy (1.3) and  $\beta = 0$ , then there exists a unique bounded continuous solution  $u^{0,\varepsilon}$  on  $\overline{\Omega}^\varepsilon \times [0, T]$  of the problem (1.4)-(1.5).

**Proof.** We consider the solution  $u^{\beta,\varepsilon}$  given by Theorem 5.1 for the choice  $u_0 = u_0^c$ . Let

$$\hat{u} = \limsup_{\beta \rightarrow 0} {}^* u^{\beta,\varepsilon} \quad \text{and} \quad \check{u} = \liminf_{\beta \rightarrow 0} {}_* u^{\beta,\varepsilon}.$$

Then, using Proposition 4.5, we obtain

$$\bar{u}^- \leq \tilde{u} \leq \hat{u} \leq \bar{u}^+ \quad \text{and} \quad |\tilde{u}|, |\hat{u}| \leq \|u_0\|_{L^\infty} + t\|F\|_{L^\infty}. \quad (5.1)$$

Using standard arguments similar to those in Step 1.1 of the proof of Theorem 1.1 in Section 6, we can show that  $\tilde{u}$  (resp.  $\hat{u}$ ) is a super- (resp. sub-) solution of the problem (1.4). The only difficulty is to recover the viscosity inequality on  $\Omega^\varepsilon \times \{0\}$ . This last inequality follows from the fact that  $u^{\beta,\varepsilon}$  is a classical solution and then satisfies the equation also at  $t = 0$ . Finally, using (5.1), we deduce that  $\tilde{u}$  (resp.  $\hat{u}$ ) is a super- (resp. sub-) solution of the problem (1.4)-(1.5). Then the comparison principle (Theorem 3.4) implies that  $\hat{u} \leq \tilde{u}$ . So  $\tilde{u} = \hat{u} =: u^{0,\varepsilon}$  is a continuous bounded solution of (1.4)-(1.5). The uniqueness of this solution follows again from the comparison principle.  $\square$

As a conclusion, there exists a unique (viscosity) solution  $u^\varepsilon \in C^0(\bar{\Omega}^\varepsilon \times [0, T])$  of problem (1.4)-(1.5) defined by:

$$u^\varepsilon := \begin{cases} u^{\beta,\varepsilon} & \text{if } \beta > 0, \\ u^{0,\varepsilon} & \text{if } \beta = 0. \end{cases} \quad (5.2)$$

**Remark 5.3** *The existence of a solution could also be proven by Perron's method.*

## 6 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Note that in the sequel, we use the following notation:

$$\Omega_T := \Omega \times (0, T), \quad \partial^l \Omega_T := \partial\Omega \times (0, T), \quad \bar{\Omega}_T := \bar{\Omega} \times [0, T],$$

and

$$\Omega_T^\varepsilon := \Omega^\varepsilon \times (0, T), \quad \partial^l \Omega_T^\varepsilon := \partial\Omega^\varepsilon \times (0, T), \quad \bar{\Omega}_T^\varepsilon := \bar{\Omega}^\varepsilon \times [0, T],$$

where  $\partial^l$  denotes the lateral boundary.

**Proof of Theorem 1.1.** The proof is divided into two steps.

**Step 1:  $\bar{u}$  and  $\underline{u}$  are, respectively, sub- and super-solution of (1.1)-(1.2).**

Let  $u^\varepsilon$  be the bounded viscosity solution of the problem (1.4)-(1.5). We define, for  $(x, t) \in \bar{\Omega}_T$ , the functions  $\bar{u}$  and  $\underline{u}$  as follows:

$$\bar{u}(x, t) := \limsup_{\substack{y \rightarrow x, s \rightarrow t, \varepsilon \rightarrow 0 \\ y \in \bar{\Omega}^\varepsilon, s \in [0, T]}} u^\varepsilon(y, s) \quad \text{and} \quad \underline{u}(x, t) := \liminf_{\substack{y \rightarrow x, s \rightarrow t, \varepsilon \rightarrow 0 \\ y \in \bar{\Omega}^\varepsilon, s \in [0, T]}} u^\varepsilon(y, s). \quad (6.1)$$

We start by showing that  $\bar{u}$  is a viscosity sub-solution of (1.1)-(1.2).

**Step 1.1: Proof of 1 (ii) in Definition 3.5.**

Let  $\varphi \in C^2(\bar{\Omega}_T)$  and  $P_0 = (x_0, t_0) \in \bar{\Omega}_T$  such that  $\bar{u} - \varphi$  has a zero local maximum at  $P_0$ . Without loss of generality, we can assume that this maximum is global and strict. Therefore we have

$$\forall r > 0, \exists \delta = \delta(r) > 0 \text{ such that } \varphi - \bar{u} \geq \delta > 0 \text{ in } \bar{\Omega}_T \setminus B_r(P_0), \quad (6.2)$$

where  $B_r(P_0) \subset \mathbb{R}^n \times \mathbb{R}$  is the open ball of radius  $r$  and of center  $P_0$ . Now, let  $w^\varepsilon := \varphi - u^\varepsilon$  for some  $\varepsilon > 0$ . Then, using the definition of  $\overline{u}$ , and inequality (6.2), we infer that

$$w^\varepsilon \geq \frac{\delta}{2} \quad \text{in } \overline{\Omega}_T^\varepsilon \setminus B_r(P_0), \quad (6.3)$$

for all  $r > 0$  and for  $\varepsilon > 0$  small enough. Using [4, Lemma 4.2], it is then classical to see that there exists a sequence  $P_\varepsilon^* = (x_\varepsilon^*, t_\varepsilon^*) \in \overline{\Omega}_T^\varepsilon \cap B_r(P_0)$  such that, as  $\varepsilon \rightarrow 0$ , we have

$$P_\varepsilon^* \rightarrow P_0, \quad u^\varepsilon(P_\varepsilon^*) \rightarrow u(P_0) \quad \text{and} \quad u^\varepsilon - \varphi \quad \text{has a local maximum at } P_\varepsilon^*. \quad (6.4)$$

Two cases are then considered.

**Case 1 :**  $P_0 \in \Omega \times [0, T)$ . We argue by contradiction. Assume that there exists a positive constant  $\gamma > 0$  such that

$$\beta \varphi_t(P_0) = \Delta \varphi(P_0) + \gamma \quad \left| \begin{array}{ll} \text{for } P_0 \in \Omega \times (0, T) & \text{if } \beta > 0 \\ \text{for } P_0 \in \Omega \times [0, T) & \text{if } \beta = 0 \end{array} \right. \quad (6.5)$$

Moreover, using (6.4) and the fact that  $u^\varepsilon$  is a viscosity sub-solution of problem (1.4)-(1.5), we conclude that

$$\beta \varphi_t(P_\varepsilon^*) \leq \Delta^\varepsilon[\varphi](P_\varepsilon^*), \quad (6.6)$$

and by Taylor's expansion, this inequality (6.6) implies

$$\beta \varphi_t(P_\varepsilon^*) \leq \Delta \varphi(P_\varepsilon^*) + o_\varepsilon(1). \quad (6.7)$$

Combining (6.5) and (6.7) yields:

$$\gamma \leq -\beta(\varphi_t(P_\varepsilon^*) - \varphi_t(P_0)) + \Delta \varphi(P_\varepsilon^*) - \Delta \varphi(P_0) + o_\varepsilon(1),$$

where the right hand side goes to zero as  $\varepsilon \rightarrow 0$ . This contradicts the fact that  $\gamma > 0$ .

**Case 2 :**  $P_0 \in \partial^l \Omega_T$ . We repeat similar arguments as in Case 1. Suppose that

$$\min \left\{ \beta \varphi_t(P_0) - \Delta \varphi(P_0), \quad \varphi_t(P_0) - F(\varphi(P_0)) - \frac{\partial \varphi}{\partial x_n}(P_0) \right\} = \gamma_1 > 0. \quad (6.8)$$

Inequality (6.8) implies

$$\beta \varphi_t(P_0) - \Delta \varphi(P_0) \geq \gamma_1, \quad (6.9)$$

and

$$\varphi_t(P_0) - F(\varphi(P_0)) - \frac{\partial \varphi}{\partial x_n}(P_0) \geq \gamma_1. \quad (6.10)$$

On the one hand, if  $P_\varepsilon^* \in \Omega_T^\varepsilon$ , then, using (6.7) and (6.9), we obtain a contradiction by using the same reasoning as in Case 1. On the other hand, if  $P_\varepsilon^* \in \partial^l \Omega_T^\varepsilon$ , then using (6.4) and the fact that  $u^\varepsilon$  is a subsolution of (1.4)-(1.5) we obtain:

$$\varphi_t(P_\varepsilon^*) \leq F(\varphi(P_\varepsilon^*)) + D^\varepsilon[\varphi](P_\varepsilon^*),$$

and then, using Taylor's expansion, we obtain

$$\varphi_t(P_\varepsilon^*) \leq \frac{\partial \varphi}{\partial x_n}(P_\varepsilon^*) + F(\varphi(P_\varepsilon^*)) + O(\varepsilon).$$

Finally, subtracting this inequality from (6.10), we conclude, after passing to the limit as  $\varepsilon \rightarrow 0$ , that  $\gamma_1 \leq 0$ ; contradiction.

**Step 1.2 : Proof of 1(i) in Definition 3.5.**

From Propositions 4.5 and 4.6, we can pass to the limit and get from the barriers

$$|\bar{u}(x, t) - u_0(x)| \leq \begin{cases} tC_\beta & \text{if } \beta > 0 \\ C(t + \sqrt{1 + x_n} - 1) & \text{if } \beta = 0 \end{cases} \quad \text{and} \quad u_0 = u_0^\varepsilon \quad \left| \quad \text{for all } (x, t) \in \bar{\Omega}_T. \right.$$

Therefore for  $t = 0$  we recover 1(i) in Definition 3.5 for all  $\beta \geq 0$ .

**Step 2: Existence and convergence.**

From Step 1 we conclude that  $\bar{u}$  is a viscosity sub-solution of (1.1)-(1.2) and by a similar manner, we can show that  $\underline{u}$  is a viscosity super-solution of (1.1)-(1.2). Moreover  $\bar{u}$  and  $\underline{u}$  are bounded, because Proposition 4.5 implies

$$|\bar{u}|, |\underline{u}| \leq \|u_0\|_{L^\infty} + t\|F\|_{L^\infty}$$

Then by the comparison principle for problem (1.1)-(1.2) (Theorem 3.6), we have  $\bar{u} \leq \underline{u}$ . On the other hand, by the definition of  $\bar{u}$  and  $\underline{u}$ , we have  $\underline{u} \leq \bar{u}$ . As a consequence, we deduce, for  $(x, t) \in \bar{\Omega}_T$ , that:

$$\bar{u}(x, t) = \underline{u}(x, t) = \lim_{\varepsilon \rightarrow 0, y \rightarrow x, s \rightarrow t} u^\varepsilon(y, s) =: u^0(x, t) \quad (6.11)$$

where  $u^0$  is then a continuous viscosity solution of problem (1.1)-(1.2). Moreover  $u^0$  is unique, still by the comparison principle. Finally, we also observe that the convergence  $u^\varepsilon \rightarrow u^0$  as  $\varepsilon \rightarrow 0$  is locally uniform in the following sense:

$$\|u^\varepsilon - u^0\|_{L^\infty(K \cap \bar{\Omega}_T^\varepsilon)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,$$

for any compact set  $K \subset \bar{\Omega}_T$ . Indeed, suppose that there exists  $\theta > 0$  such that for all  $k > 0$  there exists  $\varepsilon_k$  satisfying  $0 < \varepsilon_k < \frac{1}{k}$  with

$$\|u^{\varepsilon_k} - u^0\|_{L^\infty(K \cap (\bar{\Omega}^{\varepsilon_k} \times [0, T]))} > \theta.$$

Then, there exists a sequence  $P_k \in K \cap (\bar{\Omega}^{\varepsilon_k} \times [0, T])$  such that

$$|u^\varepsilon(P_k) - u^0(P_k)| > \theta. \quad (6.12)$$

Now, since  $K \cap (\bar{\Omega}^{\varepsilon_k} \times [0, T]) \subset K$  with  $K$  compact, there exists a subsequence, denoted for simplicity by  $P_k$ , such that  $P_k \rightarrow P_\infty$  as  $k \rightarrow \infty$ . Finally, taking the  $\liminf_{k \rightarrow \infty, P_k \rightarrow P_\infty}$  in the inequality (6.12) and using (6.11), we obtain  $0 > \theta$  which gives a contradiction.  $\square$

## 6.1 Other barriers and comments on another possible approach for $\beta = 0$

Let us notice that for  $\beta = 0$ , we have natural barriers for the  $\varepsilon$ -problem, which are:

$$u_0^{D, \varepsilon}(x) \pm C_\varepsilon t$$

where  $u_0^{D, \varepsilon} = u_0^D$  is the discrete harmonic extension (we make its dependence explicit on  $\varepsilon$ ) associated to the operator  $\Delta^\varepsilon$ , and with the constant  $C_\varepsilon$  satisfying

$$C_\varepsilon \geq D^\varepsilon[u_0^{D, \varepsilon}] + \|F\|_{L^\infty}.$$

Then another possible approach to show the convergence of  $u^\varepsilon$  to  $u^0$ , could be to control the solution as  $\varepsilon$  goes to zero, showing that:

- 1) the constant  $C_\varepsilon$  can be taken independent on  $\varepsilon$  (using the regularity of  $u_0$  on  $\partial\Omega$ ).
- 2) the discrete harmonic extension  $u_0^{D,\varepsilon}$  converges to the continuous harmonic extension  $u_0^c$  as  $\varepsilon$  tends to zero.

Then we could also introduce a (more classical) notion of viscosity solution in the case  $\beta = 0$ , assuming that at  $t = 0$ , we can compare the sub/super-solution to the initial data (taken to be equal to the harmonic extension  $u_0^{D,\varepsilon}$  for the  $\varepsilon$ -problem and  $u_0^c$  for the limit problem). Nevertheless, this other approach would require some additional work (to show 1) and 2)), and would not simplify the proofs.

**Remark 6.1 (Convergence of the discrete harmonic extension to the continuous harmonic extension)**

*Notice that point 2) is a consequence of our convergence theorem (Theorem 1.1) in the case  $\beta = 0$ . Indeed, in the case  $\beta = 0$ , the bounded solutions satisfy*

$$|u^\varepsilon(x, t) - u_0^{D,\varepsilon}(x)| \leq C_\varepsilon t$$

and then

$$u^\varepsilon(x, 0) = u_0^{D,\varepsilon}(x).$$

On the other hand, the functions

$$u_0^c(x) \pm C_0 t$$

are barriers for the limit problem if

$$C_0 \geq \left\| \frac{\partial u_0^c}{\partial n} \right\|_{L^\infty} + \|F\|_{L^\infty}.$$

Then bounded solutions of the continuous problem satisfy

$$|u^0(x, t) - u_0^c(x)| \leq C_0 t$$

and then

$$u^0(x, 0) = u_0^c(x).$$

Finally from the (locally uniform) convergence of  $u^\varepsilon$  to  $u^0$  in particular for  $t = 0$ , we deduce that  $u_0^{D,\varepsilon}$  converges, locally uniformly, to  $u_0^c$ .

## 7 Proof of Theorem 3.4

In order to emphasize the main points, we perform the proof in several steps.

**Step 1: Rescaling.** We want to reduce the problem (1.4)-(1.5) to the case  $\varepsilon = 1$  and with a nonlinearity  $F$  replaced by a monotone one. To this end, we introduce the new functions

$$\bar{u}(x, t) := e^{-\lambda t} u(\varepsilon x, \varepsilon t), \quad \bar{v}(x, t) := e^{-\lambda t} v(\varepsilon x, \varepsilon t), \quad (x, t) \in \bar{\Omega}^1 \times [0, \bar{T}),$$

where  $\bar{T} = T/\varepsilon$  and  $\lambda > 0$  is a constant to be determined later. We see easily that  $\bar{u}$  (resp.  $\bar{v}$ ) is a sub-solution (resp. super-solution) of the following problem

$$\begin{cases} \bar{\beta} \bar{u}_t = \Delta^1[\bar{u}] - \bar{\beta} \lambda \bar{u} & \text{in } \Omega^1 \times (0, \bar{T}), \\ \bar{u}_t = \bar{F}(\bar{u}, t) + D^1[\bar{u}] & \text{on } \partial\Omega^1 \times (0, \bar{T}), \end{cases} \quad (7.1)$$

and

$$\bar{u}(x, 0) = u_0(\varepsilon x) \quad \text{for} \quad \begin{cases} x \in \bar{\Omega}^1 & \text{if } \bar{\beta} > 0 \\ x \in \partial\Omega^1 & \text{if } \bar{\beta} = 0, \end{cases} \quad (7.2)$$

with

$$\bar{\beta} = \varepsilon\beta \quad \text{and} \quad \bar{F}(\bar{u}, t) = \varepsilon e^{-\lambda t} F(e^{\lambda t} \bar{u}) - \lambda \bar{u}.$$

We argue by contradiction assuming that

$$M := \sup_{\bar{\Omega}^1 \times [0, \bar{T})} (\bar{u} - \bar{v}) > 0.$$

From the definition of  $M$ , there exists a sequence  $P^k = (x^k, t^k) \in \bar{\Omega}^1 \times [0, \bar{T})$  such that  $\bar{u}(P^k) - \bar{v}(P^k) \rightarrow M > 0$  as  $k \rightarrow \infty$ . Let us choose an index  $k_0$  such that

$$\bar{u}(P^{k_0}) - \bar{v}(P^{k_0}) \geq \frac{M}{2}.$$

At this stage, if we write  $x = (x', x_n)$  with  $x' = (x_1, \dots, x_{n-1})$ , we define

$$\Psi(x, t, s) := \frac{|t - s|^2}{2\delta} + \frac{\eta}{\bar{T} - t} + \alpha|x'|^2 + \gamma\sqrt{1 + x_n}, \quad (x, t, s) \in \bar{\Omega}^1 \times [0, \bar{T})^2,$$

where  $\delta, \eta, \alpha, \gamma > 0$  will be chosen later, and we consider

$$\bar{M} := M_{\delta, \eta, \alpha, \gamma} := \sup_{x \in \bar{\Omega}^1, t, s \in [0, \bar{T})} (\bar{u}(x, t) - \bar{v}(x, s) - \Psi(x, t, s)). \quad (7.3)$$

**Step 2: A priori estimates.** Now, if we choose  $\eta, \alpha, \gamma$  such that:

$$\frac{\eta}{\bar{T} - t^{k_0}} + \alpha|(x^{k_0})'|^2 + \gamma\sqrt{1 + x_n^{k_0}} \leq \frac{M}{4},$$

we conclude that

$$\bar{M} \geq \bar{u}(P^{k_0}) - \bar{v}(P^{k_0}) - \Psi(x^{k_0}, t^{k_0}, t^{k_0}) \geq \frac{M}{4} > 0.$$

Moreover, from the definition of  $\bar{M}$ , there exists  $(\bar{x}, \bar{t}, \bar{s}) \in \bar{\Omega}^1 \times [0, \bar{T})^2$  such that

$$\bar{u}(\bar{x}, \bar{t}) - \bar{v}(\bar{x}, \bar{s}) - \Psi(\bar{x}, \bar{t}, \bar{s}) = \bar{M} > 0 \quad (7.4)$$

and

$$\frac{|\bar{t} - \bar{s}|^2}{2\delta} + \frac{\eta}{\bar{T} - \bar{t}} + \alpha|\bar{x}'|^2 + \gamma\sqrt{1 + \bar{x}_n} \leq C \quad (7.5)$$

where

$$C = \|\bar{u}\|_{L^\infty(\bar{\Omega}^1 \times [0, \bar{T}))} + \|\bar{v}\|_{L^\infty(\bar{\Omega}^1 \times [0, \bar{T}))}.$$

**Step 3: Getting contradiction.** In this step, we have to distinguish two cases:

**Case 1.** ( $\bar{t} > 0$  and  $\bar{s} > 0$ )

In this case, we have two sub-cases:

(i) If  $\bar{x} \in \Omega^1$ . From the definition of  $(\bar{x}, \bar{t}, \bar{s})$ , we see that

$$\bar{u}(\bar{x}, t) \leq \varphi^{\bar{u}}(\bar{x}, t) := \bar{M} + \bar{v}(\bar{x}, \bar{s}) + \Psi(\bar{x}, t, \bar{s}).$$

Even if  $\bar{u}$  has not the required regularity, we see by a simple approximation argument that we can choose

$$\varphi^{\bar{u}}(x, t) = \bar{u}(x, t) \quad \text{for } x \neq \bar{x}.$$

Then  $\varphi^{\bar{u}}$  is a test function for  $\bar{u}$  at  $(\bar{x}, \bar{t})$  and we deduce the following viscosity inequality

$$\bar{\beta} \frac{\eta}{(\bar{T} - \bar{t})^2} + \bar{\beta} \frac{(\bar{t} - \bar{s})}{\delta} \leq \Delta^1[\bar{u}](\bar{x}, \bar{t}) - \bar{\beta} \lambda \bar{u}(\bar{x}, \bar{t}). \quad (7.6)$$

Similarly, we have

$$\bar{v}(\bar{x}, s) \geq \varphi^{\bar{v}}(\bar{x}, s) := -\bar{M} + \bar{u}(\bar{x}, \bar{t}) - \Psi(\bar{x}, \bar{t}, s),$$

and we choose

$$\varphi^{\bar{v}}(x, s) = \bar{v}(x, s) \quad \text{for } x \neq \bar{x}.$$

We then get the viscosity inequality

$$\bar{\beta} \frac{(\bar{t} - \bar{s})}{\delta} \geq \Delta^1[\bar{v}](\bar{x}, \bar{s}) - \bar{\beta} \lambda \bar{v}(\bar{x}, \bar{s}). \quad (7.7)$$

Subtracting the two viscosity inequalities and setting  $w(x) = \bar{u}(x, \bar{t}) - \bar{v}(x, \bar{s})$ , we get

$$\begin{aligned} 0 \leq \bar{\beta} \frac{\eta}{(\bar{T} - \bar{t})^2} &\leq \Delta^1[w](\bar{x}) - \bar{\beta} \lambda w(\bar{x}) \\ &\leq \Delta^1[\bar{M} + \Psi(\cdot, \bar{t}, \bar{s})](\bar{x}) \\ &\leq \alpha \sum_{|y - \bar{x}|=1, y \in \mathbb{Z}^n} (|y'|^2 - |\bar{x}'|^2) + \gamma \sum_{|y - \bar{x}|=1, y \in \mathbb{Z}^n} (\sqrt{1 + y_n} - \sqrt{1 + \bar{x}_n}) =: A + B, \end{aligned}$$

where in the second line we have used the fact that  $w(y) - \Psi(y, \bar{t}, \bar{s}) \leq \bar{M}$  for all  $y \in \bar{\Omega}^1$  with equality for  $y = \bar{x}$ . Remark that,  $B = \gamma \zeta(\bar{x}_n)$  with, for  $a \geq 1$ ,

$$\zeta(a) = f(a+1) + f(a-1) - 2f(a) \quad \text{with } f(a) = \sqrt{1+a},$$

hence we get as in (4.12) or (4.15)

$$\zeta(a) = \sum_{\pm} \int_0^1 dt \int_0^t ds f''(a \pm s) \leq f''(a)$$

where we have used the concavity of  $f''$ . Using moreover the a priori estimate (7.5) we have  $\sqrt{1 + \bar{x}_n} \leq C/\gamma$ , and we deduce that

$$B \leq -C' \gamma^4 \quad \text{with } C' = \frac{1}{4C^3}.$$

On the other hand, we have

$$A = \alpha \sum_{|y - \bar{x}|=1, y \in \mathbb{Z}^n} (y' - \bar{x}')(y' + \bar{x}') \leq 2n\alpha(1 + 2|\bar{x}'|) \leq 2n(\alpha + 2\sqrt{\alpha}\sqrt{C}), \quad (7.8)$$

where we have used the a priori estimate (7.5) ( $\alpha|\bar{x}'|^2 \leq C$ ). Finally, if we take  $\alpha, \gamma$  small enough so that  $2n(\alpha + 2\sqrt{\alpha}\sqrt{C}) < C' \gamma^4$ , we conclude that  $0 \leq A + B < 0$ , and hence a contradiction.

(ii) If  $\bar{x} \in \partial\Omega^1$ . This is a similar case of the latter one where we use the same arguments. After the same choice of the test function, we arrive at

$$\frac{\eta}{(\bar{T} - \bar{t})^2} + \frac{\bar{t} - \bar{s}}{\delta} \leq \bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) + D^1[\bar{u}](\bar{x}, \bar{t}), \quad (7.9)$$

and

$$\frac{\bar{t} - \bar{s}}{\delta} \geq \bar{F}(\bar{v}(\bar{x}, \bar{s}), \bar{s}) + D^1[\bar{v}](\bar{x}, \bar{s}). \quad (7.10)$$

Subtracting (7.9) and (7.10), we infer that

$$\begin{aligned} \frac{\eta}{(\bar{T} - \bar{t})^2} &\leq \gamma(\sqrt{2} - 1) + \sum_{|y - \bar{x}|=1, y \in \mathbb{Z}^n, y_n \geq \bar{x}_n} \alpha(|y'|^2 - |\bar{x}'|^2) + [\bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) - \bar{F}(\bar{v}(\bar{x}, \bar{s}), \bar{s})] \\ &\leq \gamma + \sum_{|y - \bar{x}|=1, y \in \mathbb{Z}^n} \alpha(|y'|^2 - |\bar{x}'|^2) + [\bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) - \bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{s})] \\ &\quad + [\bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{s}) - \bar{F}(\bar{v}(\bar{x}, \bar{s}), \bar{s})]. \end{aligned} \quad (7.11)$$

Now, if we take  $\lambda \geq \varepsilon \|F'\|_{L^\infty(\mathbb{R})}$ , we get  $\bar{F}'_u \leq 0$  and

$$\bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{s}) - \bar{F}(\bar{v}(\bar{x}, \bar{s}), \bar{s}) \leq 0, \quad (7.12)$$

thanks to the fact that  $\bar{u}(\bar{x}, \bar{t}) - \bar{v}(\bar{x}, \bar{s}) \geq 0$ . Moreover, as  $|\bar{F}_t| \leq C_0 = C_0(\varepsilon, \lambda, \|F\|_{W^{1,\infty}(\mathbb{R})})$ , we get

$$|\bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) - \bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{s})| \leq C_0 |\bar{t} - \bar{s}| \leq C_0 \sqrt{2C\delta}, \quad (7.13)$$

where we have used the a priori estimate (7.5) ( $|\bar{t} - \bar{s}|^2/(2\delta) \leq C$ ). Now, substituting (7.8), (7.12) and (7.13) into (7.11), we infer that

$$0 < \frac{\eta}{(\bar{T} - \bar{t})^2} \leq 2n(\alpha + 2\sqrt{\alpha}\sqrt{C}) + \gamma + C_0\sqrt{2C\delta}.$$

Finally, for  $\eta > 0$  fixed, we get a contradiction by choosing  $\delta, \alpha$  and  $\gamma$  small enough.

**Case 2.** ( $\bar{t} = 0$  or  $\bar{s} = 0$ )

For  $\eta, \alpha, \gamma$  fixed, we assume that there exists a sequence  $\delta \rightarrow 0$  and  $(\bar{x}, \bar{t}, \bar{s}) = (\bar{x}^\delta, \bar{t}^\delta, \bar{s}^\delta) \in \bar{\Omega}^1 \times [0, \bar{T}]^2$  such that  $\bar{t}^\delta = 0$  or  $\bar{s}^\delta = 0$ . We deal with the case  $\bar{t}^\delta = 0$  as the other case  $\bar{s}^\delta = 0$  is similar. Then

$$\bar{u}(\bar{x}^\delta, 0) - \bar{v}(\bar{x}^\delta, \bar{s}^\delta) = \bar{M} + \Psi(\bar{x}^\delta, 0, \bar{s}) \geq \frac{M}{4} > 0.$$

From (7.5), we deduce that, up to an extraction of a subsequence, we have  $(\bar{x}^\delta, \bar{s}^\delta) \rightarrow (\bar{\bar{x}}, 0) \in \bar{\Omega}^1 \times [0, \bar{T}]$  as  $\delta \rightarrow 0$ . Therefore for  $\beta > 0$  we have

$$0 < \frac{M}{4} \leq \limsup_{\delta \rightarrow 0} (\bar{u}(\bar{x}^\delta, 0) - \bar{v}(\bar{x}^\delta, \bar{s}^\delta)) \leq \bar{u}(\bar{\bar{x}}, 0) - \bar{v}(\bar{\bar{x}}, 0) \leq 0,$$

where we have used the comparison to the initial condition. This leads to a contradiction in the case  $\beta > 0$  or  $\beta = 0$  and  $\bar{\bar{x}} \in \partial\Omega^1$ . For the case  $\beta = 0$  and  $\bar{\bar{x}} \in \Omega^1$ , we get a contradiction exactly as in case 1 i).

It is worth noticing that, in the whole proof, we first fix  $\eta$ , and then we choose respectively  $\gamma, \alpha$  and  $\delta$  small enough.  $\square$



## 8 Proof of Theorem 3.6

In this section we first present the construction of an auxiliary function  $\xi$  which plays a crucial role in the proof of Theorem 3.6.

### Lemma 8.1 (Auxiliary function)

Let  $u \in USC(\overline{\Omega} \times [0, T])$ ,  $v \in LSC(\overline{\Omega} \times [0, T])$  such that  $u \leq C_0$ ,  $v \geq -C_0$ ,  $C_0 \geq 0$ , and let  $M := \sup_{\overline{\Omega} \times [0, T]}(u - v) > 0$ . Then for all  $c > 0$ , there exists a  $C^\infty$ -function  $\xi_c : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  with  $|\xi_c| \leq 3C_0$  and a positive constant  $a_c > 0$  such that for every  $x, y \in \overline{\Omega}$  and  $s, t \in [0, T]$  :

$$v(y, s) \leq \xi_c \left( \frac{x+y}{2}, \frac{t+s}{2} \right) \leq u(x, t),$$

if  $u(x, t) - v(y, s) \geq M - a_c$ ,  $|x - y| + |t - s| \leq a_c$  and  $|x|, |y| \leq c$ .

**Proof.** We essentially revisit the proof of Lemma 5.2 in Barles [5]. Without loss of generality, we can extend the proof of our result on  $[0, T]$  by defining the end point of  $u$  and  $v$  as follows:

$$u(x, T) = \limsup_{t < T, (y, t) \rightarrow (x, T)} u(y, t), \quad v(x, T) = \liminf_{t < T, (y, t) \rightarrow (x, T)} v(y, t).$$

Let  $M := \sup_{\overline{B}_{2c} \times [0, T]}(u - v)$  with  $\overline{B}_{2c} := \{x \in \overline{\Omega}; |x| \leq 2c\}$ . Let  $\mathcal{F} = \{(x, t) \in B_{2c} \times [0, T]; u(x, t) - v(x, t) = M\}$ . We have  $u \leq C_0$ ,  $v \geq -C_0$  and  $u - v = M$  on  $\mathcal{F}$ . This implies that

$$|u|, |v| \leq C_0 + M \leq 3C_0 \quad \text{on } \mathcal{F}. \quad (8.1)$$

Moreover it is easy to check that  $\mathcal{F}$  is a closed subset of  $\overline{B}_{2c} \times [0, T]$  and that the restriction of  $u$  and  $v$  to  $\mathcal{F}$  are continuous. Therefore, the restriction to  $\mathcal{F}$  of the function  $(x, t) \mapsto (u + v)/2$  is also a continuous function on  $\mathcal{F}$  which satisfies (8.1). We may extend this function as a continuous function in  $\mathbb{R}^n \times \mathbb{R}$  (still bounded by  $3C_0$ ) and then, by standard regularization arguments, there exists a  $C^\infty$ -function  $\xi$  such that

$$\left| \tilde{\xi}(x, t) - \frac{u(x, t) + v(x, t)}{2} \right| \leq \frac{M}{8} \quad \text{on } \mathcal{F} \quad \text{and} \quad |\tilde{\xi}| \leq 3C_0 \quad \text{on } \mathbb{R}^n \times \mathbb{R}. \quad (8.2)$$

In order to show the lemma, we now argue by contradiction assuming that there exist two sequences  $(x_{a_c}, t_{a_c})$ ,  $(y_{a_c}, s_{a_c})$  such that for  $a_c$  small enough,  $u(x_{a_c}, t_{a_c}) - v(y_{a_c}, s_{a_c}) \geq M - a_c$  and  $|x_{a_c} - y_{a_c}| + |t_{a_c} - s_{a_c}| \leq a_c$  and such that, say,  $u(x_{a_c}, t_{a_c}) - \tilde{\xi} \left( \frac{x_{a_c} + y_{a_c}}{2}, \frac{t_{a_c} + s_{a_c}}{2} \right) < 0$ . Extracting, if necessary, subsequences, we may assume without loss of generality that  $(x_{a_c}, t_{a_c}), (y_{a_c}, s_{a_c}) \rightarrow (\bar{x}, \bar{t})$ . Then it is easy to show the convergence of  $u(x_{a_c}, t_{a_c}) - v(y_{a_c}, s_{a_c})$  to  $M = u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t})$ . By considering the upper semi-continuity of  $u$  and the lower semi-continuity of  $v$ , this implies, on the one hand,  $u(x_{a_c}, t_{a_c}) \rightarrow u(\bar{x}, \bar{t})$ ,  $v(y_{a_c}, s_{a_c}) \rightarrow v(\bar{x}, \bar{t})$ . On the other hand, by using the continuity of  $\tilde{\xi}$ , we obtain

$$u(\bar{x}, \bar{t}) - \tilde{\xi}(\bar{x}, \bar{t}) = \lim_{a_c \rightarrow 0} \left( u(x_{a_c}, t_{a_c}) - \tilde{\xi} \left( \frac{x_{a_c} + y_{a_c}}{2}, \frac{t_{a_c} + s_{a_c}}{2} \right) \right) \leq 0. \quad (8.3)$$

But since  $u(\bar{x}, \bar{t}) - v(\bar{x}, \bar{t}) = M$ , with  $(\bar{x}, \bar{t}) \in \mathcal{F}$ , we deduce from (8.2) that

$$\tilde{\xi}(\bar{x}, \bar{t}) \leq \frac{u(\bar{x}, \bar{t}) + v(\bar{x}, \bar{t})}{2} + \frac{M}{8} = u(\bar{x}, \bar{t}) - \frac{3M}{8} < u(\bar{x}, \bar{t}),$$

which contradicts (8.3). Finally, we arrive to the result by taking  $\xi_c$  as the restriction of  $\tilde{\xi}$  on  $[0, T)$  multiplied by a cut-off function  $\psi \in C^\infty(\mathbb{R}^n)$  such that  $\psi = 1$  on  $B_c$  and zero outside  $B_{2c}$ .  $\square$

**Proof of Theorem 3.6.** The proof is divided into three steps.

**Step 1: Test function.** In order to replace the nonlinearity  $F$  by a monotone one, we define the new functions

$$\bar{u} := e^{-\lambda t} u \quad \text{and} \quad \bar{v} := e^{-\lambda t} v,$$

where  $\lambda > 0$  is a constant which will be determined later. Obviously,  $\bar{u}$  (resp.  $\bar{v}$ ) is a sub-solution (resp. super-solution) of the following problem

$$\begin{cases} \beta \bar{u}_t = \Delta[\bar{u}] - \beta \lambda \bar{u} & \text{in } \Omega \times (0, T), \\ \bar{u}_t = \frac{\partial \bar{u}}{\partial x_n} + \bar{F}(\bar{u}, t) & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (8.4)$$

and

$$\bar{u}(x, 0) = u_0(x) \quad \text{for} \quad \begin{cases} x \in \bar{\Omega} & \text{if } \beta > 0 \\ x \in \partial\Omega & \text{if } \beta = 0, \end{cases} \quad (8.5)$$

with

$$\bar{F}(\bar{u}, t) = e^{-\lambda t} F(e^{\lambda t} \bar{u}) - \lambda \bar{u}.$$

Let us assume that  $M := \sup_{\bar{\Omega} \times [0, T)} (\bar{u} - \bar{v}) > 0$  and let us exhibit a contradiction. By the definition of the supremum, there exists  $P^{k_0} = (x^{k_0}, t^{k_0}) \in \bar{\Omega} \times [0, T)$ , for some index  $k_0$ , such that

$$\bar{u}(P^{k_0}) - \bar{v}(P^{k_0}) \geq \frac{M}{2}. \quad (8.6)$$

Let us introduce the following constant:

$$C_* = \|\bar{u}\|_{L^\infty(\bar{\Omega} \times [0, T))} + \|\bar{v}\|_{L^\infty(\bar{\Omega} \times [0, T))}.$$

We now take the following notation:  $x = (x', x_n)$  with  $x' = (x_1, \dots, x_{n-1})$ , and for  $\alpha, \gamma, \eta > 0$  (to be fixed later), we can approximate the functions  $\bar{u}, \bar{v}$  by the functions:

$$\tilde{u}(x, t) := \bar{u}(x, t) - \alpha |x'|^2 - \gamma \sqrt{1 + x_n} - \frac{\eta}{T - t} \leq C_*, \quad (8.7)$$

$$\tilde{v}(y, s) := \bar{v}(y, s) + \alpha |y'|^2 + \gamma \sqrt{1 + y_n} + \frac{\eta}{T - s} \geq -C_* \quad (8.8)$$

and define

$$\widetilde{M} := \sup_{x \in \bar{\Omega}, t \in [0, T)} (\tilde{u}(x, t) - \tilde{v}(x, t)).$$

In order to dedouble the variables in space and time, following the proof of Theorem 3.2 in [5], we define the function  $\tilde{\Phi} : (\bar{\Omega} \times [0, T))^2 \rightarrow \mathbb{R}$  by:

$$\begin{aligned} \tilde{\Phi}(x, t, y, s) &:= \frac{(x - y)^2}{\varepsilon^2} + \frac{B(x_n - y_n)^2}{\varepsilon^2} + \frac{(t - s)^2}{\delta^2} + \frac{2(t - s)(x_n - y_n)}{\delta^2} \\ &\quad - (x_n - y_n) \bar{F} \left( \xi \left( \frac{x + y}{2}, \frac{t + s}{2} \right), \frac{t + s}{2} \right), \end{aligned}$$

where the parameters  $B, \varepsilon, \delta > 0$  will be chosen later. Moreover  $c > 0$  is a constant that will be defined later (only depending on  $\alpha, \gamma, \lambda, C_*, \|F\|_{L^\infty(\mathbb{R})}$ ) and the smooth function

$$\xi = \xi_c \quad (8.9)$$

is the auxiliary function associated to  $\tilde{u}$  and  $\tilde{v}$ , and given by Lemma 8.1, which shows in particular the following estimate

$$|\xi| \leq 3C_* \quad (8.10)$$

We note that  $\tilde{\Phi}(x, t, x, t) = 0$ . Then we set

$$\overline{M} = \overline{M}_{B, \varepsilon, \delta} := \sup_{x, y \in \overline{\Omega}, t, s \in [0, T]} (\tilde{u}(x, t) - \tilde{v}(y, s) - \tilde{\Phi}(x, t, y, s)) \geq \tilde{M}. \quad (8.11)$$

**Step 2: A priori estimates.** By choosing  $\eta, \alpha, \gamma$  small enough such that

$$\frac{\eta}{T - t^{k_0}} + \alpha |(x^{k_0})'|^2 + \gamma \sqrt{1 + x_n^{k_0}} \leq \frac{M}{8},$$

it follows from (8.6) that

$$\overline{M} \geq \widetilde{M} \geq \tilde{u}(P^{k_0}) - \tilde{v}(P^{k_0}) \geq \frac{M}{4} > 0. \quad (8.12)$$

Hence, from the definition of  $\overline{M}$ , there exist sequences  $\overline{x}^k, \overline{y}^k \in \overline{\Omega}, \overline{t}^k, \overline{s}^k \in [0, T]$  such that

$$\tilde{u}(\overline{x}^k, \overline{t}^k) - \tilde{v}(\overline{y}^k, \overline{s}^k) - \tilde{\Phi}(\overline{x}^k, \overline{t}^k, \overline{y}^k, \overline{s}^k) \longrightarrow \overline{M} > 0 \quad \text{as } k \rightarrow \infty,$$

and, by taking  $k$  large enough, we deduce that

$$\tilde{u}(\overline{x}^k, \overline{t}^k) - \tilde{v}(\overline{y}^k, \overline{s}^k) - \tilde{\Phi}(\overline{x}^k, \overline{t}^k, \overline{y}^k, \overline{s}^k) \geq \frac{\overline{M}}{2} > 0. \quad (8.13)$$

From (8.10), it follows that

$$|\overline{F}(\xi(\frac{\overline{x}^k + \overline{y}^k}{2}, \frac{\overline{t}^k + \overline{s}^k}{2}), \frac{\overline{t}^k + \overline{s}^k}{2})| \leq C_1 = C_1(\lambda, C_*, \|F\|_{L^\infty(\mathbb{R})}).$$

Then if we take  $\varepsilon \leq \delta \leq 1$ , Young's inequality yields

$$|(\overline{x}_n^k - \overline{y}_n^k) \overline{F}(\xi(\frac{\overline{x}^k + \overline{y}^k}{2}, \frac{\overline{t}^k + \overline{s}^k}{2}), \frac{\overline{t}^k + \overline{s}^k}{2})| \leq \frac{(\overline{x}_n^k - \overline{y}_n^k)^2}{\varepsilon^2} + \frac{C_1^2 \varepsilon^2}{4} \leq \frac{(\overline{x}_n^k - \overline{y}_n^k)^2}{\varepsilon^2} + \frac{C_1^2 \delta^2}{4}, \quad (8.14)$$

and

$$\frac{|2(\overline{t}^k - \overline{s}^k)(\overline{x}_n^k - \overline{y}_n^k)|}{\delta^2} \leq \frac{1}{2} \frac{(\overline{t}^k - \overline{s}^k)^2}{\delta^2} + 2 \frac{(\overline{x}_n^k - \overline{y}_n^k)^2}{\varepsilon^2}. \quad (8.15)$$

Using (8.14), (8.15), we deduce that for  $B \geq 3$

$$\tilde{\Phi}(\overline{x}^k, \overline{t}^k, \overline{y}^k, \overline{s}^k) \geq \frac{(x - y)^2}{\varepsilon^2} + \frac{(B - 3)(x_n - y_n)^2}{\varepsilon^2} + \frac{(t - s)^2}{2\delta^2} - \frac{C_1^2 \delta^2}{4} \geq -\frac{C_1^2 \delta^2}{4}.$$

Using (8.13), we conclude that

$$\frac{(\overline{x}^k - \overline{y}^k)^2}{\varepsilon^2} + \frac{(\overline{t}^k - \overline{s}^k)^2}{2\delta^2} + \alpha (|(\overline{x}^k)'|^2 + |(\overline{y}^k)'|^2) + \gamma \sqrt{1 + \overline{x}_n^k} + \gamma \sqrt{1 + \overline{y}_n^k} + \frac{\eta}{T - \overline{t}^k} + \frac{\eta}{T - \overline{s}^k} \leq C_{**},$$

with (for  $\delta \leq 1$ )

$$C_{**} = C_* + \frac{C_1^2}{4} = C_{**}(\lambda, C_*, \|F\|_{L^\infty(\mathbb{R})}).$$

Then, up to an extraction of a subsequence, we have

$$(\bar{x}^k, \bar{t}^k, \bar{y}^k, \bar{s}^k) \rightarrow (\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in (\bar{\Omega})^2 \times [0, T]^2 \quad \text{as } k \rightarrow \infty,$$

with

$$\tilde{u}(\bar{x}, \bar{t}) - \tilde{v}(\bar{y}, \bar{s}) - \tilde{\Phi}(\bar{x}, \bar{t}, \bar{y}, \bar{s}) = \bar{M} > 0 \quad \text{and} \quad \tilde{\Phi}(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \geq -\frac{C_1^2 \delta^2}{4} \quad (8.16)$$

and the same a priori estimate

$$\frac{(\bar{x} - \bar{y})^2}{\varepsilon^2} + \frac{(\bar{t} - \bar{s})^2}{2\delta^2} + \alpha(|\bar{x}'|^2 + |\bar{y}'|^2) + \gamma\sqrt{1 + \bar{x}_n} + \gamma\sqrt{1 + \bar{y}_n} + \frac{\eta}{T - \bar{t}} + \frac{\eta}{T - \bar{s}} \leq C_{**}. \quad (8.17)$$

**Step 3: Getting contradiction.** In order to get a contradiction, we have to distinguish two cases:

**Case 1.** ( $\bar{t} > 0$  and  $\bar{s} > 0$ )

(i) **If  $\bar{x} \in \partial\Omega$  or  $\bar{y} \in \partial\Omega$ .** We only study the case  $\bar{x} \in \partial\Omega$  as it is similar for  $\bar{y} \in \partial\Omega$ . Let  $\varphi^{\bar{u}} : \bar{\Omega} \times [0, T) \rightarrow \mathbb{R}$  be the function defined by

$$\varphi^{\bar{u}}(x, t) := \bar{M} + \tilde{v}(\bar{y}, \bar{s}) + \tilde{\Phi}(x, t, \bar{y}, \bar{s}) + \alpha|x'|^2 + \gamma\sqrt{1 + x_n} + \frac{\eta}{T - t}.$$

Using (8.16), we see that  $\varphi^{\bar{u}}$  is a test function for  $\bar{u}$  and hence we obtain

$$\min \left\{ \beta\varphi_t^{\bar{u}}(P_0) - \Delta\varphi^{\bar{u}}(P_0) - \beta\lambda\bar{u}, \varphi_t^{\bar{u}}(P_0) - \bar{F}(\varphi^{\bar{u}}(P_0), \bar{t}) - \frac{\partial\varphi^{\bar{u}}}{\partial x_n}(P_0) \right\} \leq 0,$$

where  $P_0 = (\bar{x}, \bar{t})$ . We note that the case  $\beta\varphi_t^{\bar{u}}(P_0) - \Delta\varphi^{\bar{u}}(P_0) - \beta\lambda\bar{u} \leq 0$  can be treated as in the sub-case (ii) below, so it is sufficient to study the case where  $\varphi_t^{\bar{u}}(P_0) - \bar{F}(\varphi^{\bar{u}}(P_0), \bar{t}) - \frac{\partial\varphi^{\bar{u}}}{\partial x_n}(P_0) \leq 0$ , which implies, with  $\bar{x}_n = 0$ :

$$\begin{aligned} & \frac{\eta}{(T - \bar{t})^2} + \frac{2(\bar{x}_n - \bar{y}_n)}{\delta^2} - \frac{\gamma}{2\sqrt{1 + \bar{x}_n}} + \bar{F}\left(\xi\left(\frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2}\right), \frac{\bar{t} + \bar{s}}{2}\right) - \bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) \\ & - \frac{2(1 + B)(\bar{x}_n - \bar{y}_n)}{\varepsilon^2} - \frac{(\bar{x}_n - \bar{y}_n)}{2}(\bar{F})'_u \xi'_t + \frac{(\bar{x}_n - \bar{y}_n)}{2}(\bar{F})'_u \xi'_x - \frac{(\bar{x}_n - \bar{y}_n)}{2}(\bar{F})'_t \leq 0, \end{aligned} \quad (8.18)$$

where

$$\xi'_t = \xi'_t\left(\frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2}\right), \quad (\bar{F})'_t = (\bar{F})'_t\left(\xi\left(\frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2}\right), \frac{\bar{t} + \bar{s}}{2}\right) \quad (8.19)$$

and

$$\xi'_x = \xi'_x\left(\frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2}\right), \quad (\bar{F})'_u = (\bar{F})'_u\left(\xi\left(\frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2}\right), \frac{\bar{t} + \bar{s}}{2}\right) \quad (8.20)$$

are the first partial derivatives of  $\xi$  and  $\bar{F}$ . On the one hand, from (8.16) and (8.11), we deduce that

$$\tilde{u}(\bar{x}, \bar{t}) - \tilde{v}(\bar{y}, \bar{s}) \geq \tilde{M} - \frac{C_1^2 \delta^2}{4}. \quad (8.21)$$

From (8.17) we deduce that

$$|\bar{x}|, |\bar{y}| \leq C_2 = C_2(\alpha, \gamma, C_{**}) \quad \text{and} \quad |\bar{x} - \bar{y}| \leq \varepsilon \sqrt{C_{**}}, \quad |\bar{t} - \bar{s}| \leq \delta \sqrt{2C_{**}}. \quad (8.22)$$

Then in (8.9), we can choose

$$c = c(\alpha, \gamma, \lambda, C_*, \|F\|_{L^\infty(\mathbb{R})}) := C_2$$

and Lemma 8.1 gives the existence of a number  $a_c > 0$ . Then choosing  $0 < \varepsilon \leq \delta$  small enough, we deduce from (8.21) and (8.22) that

$$\tilde{u}(\bar{x}, \bar{t}) - \tilde{v}(\bar{y}, \bar{s}) \geq \tilde{M} - a_c, \quad |\bar{x} - \bar{y}| + |\bar{t} - \bar{s}| \leq a_c \quad \text{and} \quad |\bar{x}|, |\bar{y}| \leq c.$$

Therefore, we deduce from Lemma 8.1 that

$$\xi \left( \frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2} \right) \leq \tilde{u}(\bar{x}, \bar{t}) \leq \bar{u}(\bar{x}, \bar{t}).$$

On the other hand, for the choice  $\lambda \geq \|F'\|_{L^\infty(\mathbb{R})}$ , we have  $\bar{F}'_u \leq 0$  and then we can estimate

$$\bar{F} \left( \xi \left( \frac{\bar{x} + \bar{y}}{2}, \frac{\bar{t} + \bar{s}}{2} \right), \frac{\bar{t} + \bar{s}}{2} \right) - \bar{F}(\bar{u}(\bar{x}, \bar{t}), \bar{t}) \geq -C|\bar{t} - \bar{s}|. \quad (8.23)$$

Now, as the terms in (8.19) and (8.20) are bounded, we conclude from (8.18), (8.23) and (8.17) that

$$\frac{\eta}{T^2} - \frac{\gamma}{2} \leq O\left(\frac{\varepsilon}{\delta^2}\right) + O(\varepsilon) + O(\delta).$$

Moreover, for  $\eta > 0$  fixed, we get the contradiction if we take  $\varepsilon, \delta, \gamma$  small enough so that  $\varepsilon = \varepsilon(\delta) \leq \delta^3 \leq 1$  and  $\gamma < \eta/T^2$ . We note that in the case of  $(\bar{x}, \bar{y}) \in \Omega \times \partial\Omega$ , the test function for  $\bar{v}$  is given by

$$\varphi^{\bar{v}}(y, s) := -\bar{M} + \tilde{u}(\bar{x}, \bar{t}) - \tilde{\Phi}(\bar{x}, \bar{t}, y, s) - \alpha|y'|^2 - \gamma\sqrt{1 + y_n} - \frac{\eta}{T - s}.$$

**(ii) If  $\bar{x} \in \Omega$  and  $\bar{y} \in \Omega$ .** We know from (8.16) that  $\bar{u}(\bar{x}, \bar{t}) - \bar{v}(\bar{y}, \bar{s}) - \Phi(\bar{x}, \bar{t}, \bar{y}, \bar{s})$  has a local maximum at  $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$ , where

$$\Phi(\bar{x}, \bar{t}, \bar{y}, \bar{s}) := \tilde{\Phi}(\bar{x}, \bar{t}, \bar{y}, \bar{s}) + \alpha(|\bar{x}'|^2 + |\bar{y}'|^2) + \gamma \left( \sqrt{1 + \bar{x}_n} + \sqrt{1 + \bar{y}_n} \right) + \frac{\eta}{T - \bar{t}} + \frac{\eta}{T - \bar{s}}.$$

Then it is natural to apply the classical Ishii's Lemma in the elliptic case with the new coordinates  $(\tilde{x} = (x, t), \tilde{y} = (y, s))$  and this is what we do. Indeed, we only use a corollary of Ishii's Lemma, namely Corollary 9.3 which is given in the Appendix. Applying Corollary 9.3, we get for every  $\mu > 0$  satisfying  $\mu \bar{A} < \bar{I}$ , (with  $\bar{A}$  defined in (9.2) and  $\bar{I}$  is the identity matrix of  $\mathbb{R}^{2(n+1)}$ ), the existence of symmetric  $n \times n$  matrices  $X, Y$  such that

$$(\tau_1, p_1, \begin{pmatrix} X & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^+ \bar{u}(\bar{x}, \bar{t}), \quad (\tau_2, p_2, \begin{pmatrix} Y & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^- \bar{v}(\bar{y}, \bar{s}), \quad (8.24)$$

and

$$-\left(\frac{1}{\mu} + \|\bar{A}\|\right) \hat{I} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + 2\mu(A^2 + A_1 \cdot A_1^T), \quad (8.25)$$

where  $\hat{I}$  is the identity matrix of  $\mathbb{R}^{2n}$  and where all the quantities are computed at the point  $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ :  $\tau_1 := \partial_t \Phi$ ,  $p_1 := D_x \Phi$ ,  $\tau_2 := -\partial_s \Phi$ ,  $p_2 := -D_y \Phi$ , and

$$A = \begin{pmatrix} D_{xx}^2 \Phi & D_{xy}^2 \Phi \\ D_{yx}^2 \Phi & D_{yy}^2 \Phi \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} D_{xt}^2 \Phi & D_{xs}^2 \Phi \\ D_{yt}^2 \Phi & D_{ys}^2 \Phi \end{pmatrix}.$$

A simple computation shows that

$$\begin{aligned} A &= \frac{2}{\varepsilon^2} \left( \begin{array}{c|c} I & -I \\ \hline -I & I \end{array} \right) + \frac{2B}{\varepsilon^2} \left( \begin{array}{c|c} I_n & -I_n \\ \hline -I_n & I_n \end{array} \right) + 2\alpha \left( \begin{array}{c|c} I' & 0 \\ \hline 0 & I' \end{array} \right) - \frac{\gamma}{4} \left( \begin{array}{c|c} \frac{1}{(1+\bar{x}_n)^{3/2}} I_n & 0 \\ \hline 0 & \frac{1}{(1+\bar{y}_n)^{3/2}} I_n \end{array} \right) \\ &\quad - \left( \begin{array}{c|c} \begin{pmatrix} 0 & & P' \\ \hline & 0 & \\ \hline P' & & 2P_n \end{pmatrix} & \begin{pmatrix} 0 & & -P' \\ \hline & 0 & \\ \hline P' & & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & & P' \\ \hline & 0 & \\ \hline -P' & & 0 \end{pmatrix} & \begin{pmatrix} 0 & & -P' \\ \hline & 0 & \\ \hline -P' & & -2P_n \end{pmatrix} \end{array} \right) - (\bar{x}_n - \bar{y}_n) \left( \begin{array}{c|c} D^2 \tilde{F} & D^2 \tilde{F} \\ \hline D^2 \tilde{F} & D^2 \tilde{F} \end{array} \right). \end{aligned}$$

Here  $I$  is the identity matrix of  $\mathbb{R}^n$  and for all  $i, j \in \{1, \dots, n\}$ ,  $(I_n)_{i,j} = 1$  if  $i = j = n$  and 0 otherwise, and  $(I')_{i,j} := 1$  if  $i = j \leq n-1$  and 0 otherwise. Moreover

$$\tilde{F}(x, y) := \bar{F}(\xi(\frac{\bar{t} + \bar{s}}{2}, \frac{x + y}{2}), \frac{\bar{t} + \bar{s}}{2}), \quad P' := D_{x'} \tilde{F} = D_{y'} \tilde{F}, \quad P_n := D_{x_n} \tilde{F} = D_{y_n} \tilde{F},$$

for all  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  and  $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ . Then we can write the viscosity inequalities for the limit sub/superdifferentials  $\bar{D}^- \bar{v}(\bar{y}, \bar{s})$  and  $\bar{D}^+ \bar{u}(\bar{x}, \bar{t})$ . This gives

$$\beta \tau_1 + \beta \lambda \bar{u}(\bar{x}, \bar{t}) - \text{tr}(X) \leq 0$$

and

$$\beta \tau_2 + \beta \lambda \bar{v}(\bar{y}, \bar{s}) - \text{tr}(Y) \geq 0,$$

where  $\text{tr}$  is the trace of the appropriate matrix. Subtracting these viscosity inequalities and using (8.16), we get

$$\beta(\partial_t \Phi + \partial_s \Phi) + \beta \lambda(\Phi + \bar{M}) \leq \text{tr}(X - Y),$$

i.e.

$$\beta \left( \frac{\eta}{(T - \bar{t})^2} + \frac{\eta}{(T - \bar{s})^2} - (\bar{x}_n - \bar{y}_n)((\bar{F})'_t + (\bar{F})'_u \xi'_t) \right) + \beta \lambda(\Phi + \bar{M}) \leq \text{tr}(X - Y).$$

Moreover, using the definition of  $\Phi$ , we conclude that

$$\begin{aligned} &\beta \left( \frac{\eta}{(T - \bar{t})^2} + \frac{\eta}{(T - \bar{s})^2} - (\bar{x}_n - \bar{y}_n)((\bar{F})'_t + (\bar{F})'_u \xi'_t) \right) + \beta \lambda(\bar{M} - (\bar{x}_n - \bar{y}_n) \bar{F}) \\ &\quad + \beta \lambda \left( \frac{B(\bar{x}_n - \bar{y}_n)^2}{\varepsilon^2} + \frac{(\bar{t} - \bar{s})^2}{\delta^2} + \frac{2(\bar{t} - \bar{s})(\bar{x}_n - \bar{y}_n)}{\delta^2} \right) \\ &\leq \text{tr}(X - Y). \end{aligned} \tag{8.26}$$

Using the fact that  $B \geq 1$  and  $\bar{M} > 0$ , we deduce that

$$2\beta \frac{\eta}{T^2} - \beta(\bar{x}_n - \bar{y}_n)((\bar{F})'_t + (\bar{F})'_u \xi'_t + \lambda \bar{F}) \leq \text{tr}(X - Y).$$

From (8.22), we deduce that

$$2\beta \frac{\eta}{T^2} \leq \text{tr}(X - Y) + O(\varepsilon). \tag{8.27}$$

On the other hand, taking the matrix inequality (8.25) on the vectors  $\begin{pmatrix} \xi \\ \xi \end{pmatrix}^T$  and  $\begin{pmatrix} \xi \\ \xi \end{pmatrix}$ , we get

$$\xi^T(X - Y)\xi \leq K_1 + K_2$$

with

$$K_1 = \begin{pmatrix} \xi \\ \xi \end{pmatrix}^T A \begin{pmatrix} \xi \\ \xi \end{pmatrix} = 4\alpha|\xi'|^2 - \frac{\gamma}{4} \left( \frac{1}{(1 + \bar{x}_n)^{\frac{3}{2}}} + \frac{1}{(1 + \bar{y}_n)^{\frac{3}{2}}} \right) \xi_n^2 - 4(\bar{x}_n - \bar{y}_n)\xi^T D^2 \tilde{F} \xi$$

and

$$K_2 = 2\mu \begin{pmatrix} \xi \\ \xi \end{pmatrix}^T (A^2 + A_1 \cdot A_1^T) \begin{pmatrix} \xi \\ \xi \end{pmatrix} \leq 4\mu \|A^2 + A_1 \cdot A_1^T\| |\xi|^2.$$

Using again the a priori estimate (8.17), we have

$$-\frac{\gamma}{4} \left\{ \frac{1}{(1 + \bar{x}_n)^{3/2}} + \frac{1}{(1 + \bar{y}_n)^{3/2}} \right\} \leq -C'\gamma^4,$$

and we conclude that

$$\text{tr}(X - Y) \leq 4\alpha(n - 1) - C'\gamma^4 - 4(\bar{x}_n - \bar{y}_n)\text{tr}(D^2 \tilde{F}) + 4\mu n \|A^2 + A_1 \cdot A_1^T\|. \quad (8.28)$$

From (8.27) and the bounds on  $F$  in  $W^{2,\infty}(\mathbb{R})$ , we deduce that

$$2\beta \frac{\eta}{T^2} \leq 4\alpha(n - 1) - C'\gamma^4 + O(\varepsilon) + 4\mu n \|A^2 + A_1 \cdot A_1^T\|.$$

Because all the terms are independent on  $\mu$ , we can first take the limit  $\mu \rightarrow 0$ . Recall that  $\beta \geq 0$ , and then we can not use  $\eta$  to get a contradiction as usual. We then have to get a contradiction only using the following inequality:

$$0 \leq 4\alpha(n - 1) - C'\gamma^4 + O(\varepsilon).$$

To this end, for given  $\gamma > 0$ , we get a contradiction choosing  $\alpha, \varepsilon$  small enough such that  $\varepsilon \ll \gamma^4$  and  $8\alpha(n - 1) < C'\gamma^4$ .

**Case 2.** ( $\bar{t} = 0$  or  $\bar{s} = 0$ )

By fixing the parameters  $\gamma, \eta$  and  $\alpha$ , we assume that there exists a sequence  $\delta \rightarrow 0$  and  $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) = (\bar{x}^\delta, \bar{t}^\delta, \bar{y}^\delta, \bar{s}^\delta) \in (\bar{\Omega} \times [0, T])^2$  such that  $\bar{s}^\delta = 0$  or  $\bar{t}^\delta = 0$ . It is sufficient to study the case  $\bar{t}^\delta = 0$  as the other case can be deduced similarly. Thus

$$\tilde{u}(\bar{x}^\delta, 0) - \tilde{v}(\bar{y}^\delta, \bar{s}^\delta) = \bar{M} + \tilde{\Phi}(\bar{x}^\delta, 0, \bar{y}^\delta, \bar{s}^\delta),$$

and we deduce from (8.17) (with  $\varepsilon \leq \delta^3 \leq 1$ ) that, up to extraction of a subsequence, we have  $(\bar{x}^\delta, 0, \bar{y}^\delta, \bar{s}^\delta) \rightarrow (\bar{x}, 0, \bar{y}, 0) \in (\bar{\Omega} \times [0, T])^2$  as  $\delta \rightarrow 0$ . Hence in the case  $\beta > 0$ , we conclude that (using (8.16))

$$0 < \bar{M} \leq \limsup_{\delta \rightarrow 0} (\bar{M} + \tilde{\Phi}(\bar{x}^\delta, 0, \bar{y}^\delta, \bar{s}^\delta)) = \limsup_{\delta \rightarrow 0} (\tilde{u}(\bar{x}^\delta, 0) - \tilde{v}(\bar{y}^\delta, \bar{s}^\delta)) \leq \bar{u}(\bar{x}, 0) - \bar{v}(\bar{x}, 0) \leq 0,$$

where we have used the comparison to the initial condition. This gives a contradiction in the case  $\beta > 0$  and in the case  $\beta = 0$  if  $\bar{x} \in \partial\Omega$ . In the case  $\beta = 0$  and  $\bar{x} \in \Omega$ , we get a contradiction exactly as in the case 1 i).

In the whole proof, we first fix  $\eta$  and  $B$ . After that, we fix respectively  $\gamma, \alpha, \delta, \varepsilon$  (and  $\mu$ ).  $\square$

## 9 Appendix

This appendix is dedicated to the proof of a corollary of Ishii's lemma (Corollary 9.3), which is used in the proof of Theorem 3.6. First, we recall the elliptic sub and superdifferentials of semi-continuous functions and the classical Ishii's Lemma. In all what follows, we denote by  $\mathbb{S}^n$  the set of symmetric  $n \times n$  matrices.

### Definition 9.1 (Elliptic sub and superdifferential of order two)

Let  $U$  be a locally compact subset of  $\mathbb{R}^n$  and  $u \in USC(U)$ . Then the superdifferential  $\mathcal{D}^+u$  of order two of the function  $u$  is defined by:  $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$  belongs to  $\mathcal{D}^+u(x)$  if  $x \in U$  and

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$$

as  $U \ni y \rightarrow x$ . In a similar way, we define the subdifferential of order two by  $\mathcal{D}^-u = -\mathcal{D}^+(-u)$ . We also define:

$$\overline{\mathcal{D}}^+u(x) = \left\{ \begin{array}{l} (p, X) \in \mathbb{R}^n \times \mathbb{S}^n, \exists (x_n, p_n, X_n) \in U \times \mathbb{R}^n \times \mathbb{S}^n \\ \text{such that } (p_n, X_n) \in \mathcal{D}^+u(x_n) \\ \text{and } (x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X) \end{array} \right\}.$$

The set  $\overline{\mathcal{D}}^-u(x)$  is defined in a similar way.

Now recall the classical elliptic version of Ishii's Lemma.

### Lemma 9.2 (Elliptic version of Ishii's lemma)

Let  $U$  and  $V$  be a locally compact subsets of  $\mathbb{R}^n$ ,  $u \in USC(U)$  and  $v \in LSC(V)$ . Let  $\varphi : U \times V \rightarrow \mathbb{R}$  be of class  $C^2$ . Assume that  $(x, y) \mapsto u(x) - v(y) - \varphi(x, y)$  reaches a local maximum at  $(\bar{x}, \bar{y}) \in U \times V$ . We note  $p_1 = D_x \varphi(\bar{x}, \bar{y})$ ,  $p_2 = -D_y \varphi(\bar{x}, \bar{y})$  and  $A = D^2 \varphi(\bar{x}, \bar{y})$ . Then, for every  $\mu > 0$  such that  $\mu A < \hat{I}$ , there exists  $X, Y \in \mathbb{S}^n$  such that:

$$\begin{aligned} (p_1, X) &\in \overline{\mathcal{D}}^+u(\bar{x}), \quad (p_2, Y) \in \overline{\mathcal{D}}^-v(\bar{y}), \\ -\left(\frac{1}{\mu} + \|A\|\right)\hat{I} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \mu A^2, \end{aligned} \tag{9.1}$$

where  $\hat{I}$  is the identity matrix of  $\mathbb{R}^{2n}$ . The norm of the symmetric matrix  $A$  used in (9.1) is

$$\|A\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} = \sup\{|\langle A\xi, \xi \rangle| : |\xi| \leq 1\}.$$

For the proof, we refer the reader to Theorem 3.2 in the User's Guide [15].

Then we have:

### Corollary 9.3 (Consequence of Ishii's lemma)

Given  $T > 0$ . Let  $U$  and  $V$  be a locally compact subsets of  $\mathbb{R}^n$ ,  $u \in USC(U \times [0, T])$  and  $v \in LSC(V \times [0, T])$ . Let  $\Phi : U \times [0, T] \times V \times [0, T] \rightarrow \mathbb{R}$  be of class  $C^2$ . Assume that  $(x, t, y, s) \mapsto u(x, t) - v(y, s) - \Phi(x, t, y, s)$  reaches a local maximum in  $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in U \times [0, T] \times V \times [0, T]$ . Computing the following quantities at the point  $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ , we set  $\tau_1 = \partial_t \Phi$ ,  $\tau_2 = -\partial_s \Phi$ ,  $p_1 = D_x \Phi$ ,  $p_2 = -D_y \Phi$  and

$$\overline{A} = \begin{pmatrix} A & A_1 \\ A_1^T & A_2 \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} D_{xx}^2 \Phi & D_{xy}^2 \Phi \\ D_{yx}^2 \Phi & D_{yy}^2 \Phi \end{pmatrix}, \quad A_1 = \begin{pmatrix} D_{xt}^2 \Phi & D_{xs}^2 \Phi \\ D_{yt}^2 \Phi & D_{ys}^2 \Phi \end{pmatrix}, \quad A_2 = \begin{pmatrix} D_{tt}^2 \Phi & D_{ts}^2 \Phi \\ D_{st}^2 \Phi & D_{ss}^2 \Phi \end{pmatrix}. \tag{9.2}$$



Let  $\bar{I}$  be the identity matrix of  $\mathbb{R}^{2(n+1)}$ . Then for every  $\mu > 0$  such that  $\mu\bar{A} < \bar{I}$ , there exists  $X, Y \in \mathcal{S}^n$  such that (where we denote by  $*$  some elements that we do not precise):

$$(\tau_1, p_1, \begin{pmatrix} X & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^+ \bar{u}(\bar{x}, \bar{t}), \quad (\tau_2, p_2, \begin{pmatrix} Y & * \\ * & * \end{pmatrix}) \in \bar{\mathcal{D}}^- \bar{v}(\bar{y}, \bar{s}),$$

and

$$-(\frac{1}{\mu} + \|\bar{A}\|)\hat{I} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + 2\mu(A^2 + A_1 \cdot A_1^T).$$

where  $\hat{I}$  is the identity matrix of  $\mathbb{R}^{2n}$ .

**Proof.** Because  $\Phi \in C^2$  and  $u(x, t) - v(y, s) - \Phi(x, t, y, s)$  admits a local maximum in  $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ , we can then apply the elliptic Ishii's Lemma (Lemma 9.2) with the new variables  $\tilde{x} := (\bar{x}, \bar{t})$  and  $\tilde{y} := (\bar{y}, \bar{s})$ . We obtain, for every  $\mu$  satisfying  $\mu\tilde{A} < \bar{I}$  with the  $2(n+1) \times 2(n+1)$  matrix  $\tilde{A} = D^2\Phi$ , that there exists  $\tilde{X}, \tilde{Y} \in \mathcal{S}^{n+1}$  with

$$\tilde{X} = \begin{pmatrix} X & C \\ {}^tC & \delta \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} Y & D \\ {}^tD & \rho \end{pmatrix} \quad \text{with} \quad X, Y \in \mathcal{S}^n, \quad C, D \in \mathbb{R}^n \quad \text{and} \quad \delta, \rho \in \mathbb{R}$$

such that:

$$((p_1, \tau_1), \tilde{X}) \in \bar{\mathcal{D}}^+ u(\bar{x}, \bar{t}), \quad ((p_2, \tau_2), \tilde{Y}) \in \bar{\mathcal{D}}^- v(\bar{y}, \bar{s})$$

and

$$-(\frac{1}{\mu} + \|\tilde{A}\|)\bar{I} \leq \left( \begin{array}{cc|cc} X & C & & 0 \\ {}^tC & \delta & & \\ \hline & & -Y & -D \\ 0 & & -{}^tD & -\rho \end{array} \right) \leq \tilde{A} + \mu\tilde{A}^2. \quad (9.3)$$

We first remark that the matrix  $\bar{A}$  (defined in (9.2)) is obtained from  $\tilde{A}$  by relabeling the vectors of the basis (going from coordinates  $(x, t, y, s)$  for  $\tilde{A}$  to coordinates  $(x, y, t, s)$  for  $\bar{A}$ ). Therefore we have  $\|\tilde{A}\| = \|\bar{A}\|$  and the condition  $\mu\tilde{A} < \bar{I}$  is equivalent to  $\mu\bar{A} < \bar{I}$ .

Next, for  $\xi, \eta \in \mathbb{R}^n$ , applying the vector  $\tilde{V} = (\xi, 0, \eta, 0)^T$  to the matrix inequality (9.3), yields with  $V = (\xi, \eta)^T$ :

$$\begin{aligned} -(\frac{1}{\mu} + \|\bar{A}\|) \langle \hat{I} \cdot V, V \rangle &\leq \langle \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \cdot V, V \rangle \\ &\leq \langle (\tilde{A} + \mu\tilde{A}^2) \cdot \tilde{V}, \tilde{V} \rangle =: J. \end{aligned} \quad (9.4)$$

Remark that using the relabeling of the vectors of the basis, we have

$$\langle \tilde{A}^k \tilde{V}, \tilde{V} \rangle = \langle \bar{A}^k \bar{V}, \bar{V} \rangle \quad \text{with} \quad \bar{V} = \begin{pmatrix} V \\ 0_{\mathbb{R}^2} \end{pmatrix}, \quad \text{for} \quad k = 1, 2$$

We compute

$$\begin{pmatrix} V \\ 0 \end{pmatrix}^T \begin{pmatrix} A & A_1 \\ A_1^T & A_2 \end{pmatrix} \begin{pmatrix} V \\ 0 \end{pmatrix} = V^T A V$$

and

$$\begin{pmatrix} V \\ 0 \end{pmatrix}^T \begin{pmatrix} A & A_1 \\ A_1^T & A_2 \end{pmatrix}^2 \begin{pmatrix} V \\ 0 \end{pmatrix} = V^T A^2 V + V^T A_1 A_1^T V + 2 \langle AV, A_1^T V \rangle \leq 2(|AV|^2 + |A_1^T V|^2)$$

This gives

$$J \leq V^T AV + 2\mu (|AV|^2 + |A_1^T V|^2)$$

and then (9.4) implies

$$-(\frac{1}{\mu} + \|\overline{A}\|)\hat{I} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + 2\mu(A^2 + A_1 \cdot A_1^T),$$

which achieves the proof.  $\square$

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